# Making Advanced Mathematics Work in Secondary Teacher Education

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We argue that it is essential to provide opportunities for prospective secondary mathematics teachers to connect advanced mathematics content to secondary mathematics teaching practice (in addition to connections only to secondary mathematics), if advanced mathematics courses are to be useful to these teachers. In light of this argument, we review a selection of curricular materials satisfying two criteria: first, they are written for use in advanced mathematics courses that prospective teachers may take; and second, they feature explicit connections to secondary teaching practice. We use this review to highlight essential questions for the future of research and practice in advanced mathematics coursework. We suggest there is much unknown about how teachers can successfully integrate their experiences in advanced mathematics and pedagogical methods.

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## Introduction

In many university-based secondary mathematics teacher certification programs in the United States (U.S.), teachers complete multiple courses in advanced mathematics (e.g., Ferrini-Mundy & Findell, 2004; Tatto & Bankov, 2018). Our purposes regarding advanced mathematics coursework are twofold. First, we argue that it is essential to provide opportunities for prospective secondary teachers to connect the content of these courses to the *practice of teaching*, if these courses are to be useful to prospective teachers. Second, we suggest there is much unknown about how prospective teachers can successfully integrate course experiences in advanced mathematics and pedagogical methods. This is a critical issue, especially if instruction in both seeks to shape teachers' images of teaching practice. Our arguments implicate AMTE indicators P.2.1, P.2.2, P.3.1, and P.3.4 (Association of Mathematics Teacher Educators, 2017)[i], and how well teachers can connect experiences across these standards.

Note that throughout this chapter, we use "teacher" to refer to "prospective secondary mathematics teacher," unless otherwise stated. We use "instructor" or "faculty" to refer to those teaching university-level courses.

Advanced mathematics content and practice can illuminate secondary mathematics content and practice (e.g., Baldinger, 2018; Lai & Donsig, 2018; Wasserman & Weber, 2017). Yet despite this relationship, exemplifying ties between advanced and secondary mathematics, let alone integrating advanced mathematics and the practice of secondary teaching, has been challenging (e.g.,Wasserman & Weber, 2017; Yan et al., 2022). These connections are "not currently well developed in the profession of mathematics teaching" (CBMS, 2012, p. 14). In this chapter, we:

- Demonstrate that, historically, teachers' pedagogy does not usually improve, or even change, as a result of taking advanced mathematics coursework;
- Synthesize the recent movement to connect advanced mathematics coursework to secondary mathematics *teaching practice*, and why these opportunities are important to provide; and
- Pose essential questions for the future of advanced mathematics courses in secondary mathematics teacher education, especially in view of enabling teachers to integrate experiences in mathematics and methods coursework.

Although the field has made initial progress, we have much to learn about providing teachers robust opportunities to connect advanced mathematics and secondary teaching practice.

### Advanced Mathematics Courses Should Do More to Connect to Secondary Teaching

Advanced mathematics knowledge can potentially strengthen secondary mathematics knowledge (AMTE SPTM Indicator P.2.1, 2017; CBMS, 2012). Historically, a typical approach has been to follow Felix Klein's (1924/1932) prescription of considering "elementary mathematics from an advanced standpoint" (p. 1). The hope here is that with a more sophisticated understanding of secondary concepts, teachers will take more productive actions when teaching.[ii] The challenge, though, is that how this knowledge influences teaching is less clear–what can or should secondary teachers conceive or do differently as a result of this strengthened knowledge?

This lack of clarity impacts teachers, who lament their mathematical preparation was irrelevant to secondary teaching (e.g., Goulding et al., 2003; Ticknor, 2012; Wasserman & Galarza, 2018; Zazkis & Leikin, 2010). Even when prospective teachers develop a richer understanding of the secondary mathematical concepts that they will teach, they may still exit advanced mathematics believing that they may have become better mathematicians, but not better mathematics teachers (Wasserman & Ham, 2013). Regardless, while many teachers may develop stronger personal mathematical practices through advanced coursework (Indicator P.2.2), they still may not see why it matters that mathematics can be deep and meaningful (Indicator P.3.1), nor how their mathematical growth might shape their pedagogical growth (Indicator P.3.4).

## Reflecting on the Past: Connections Through *Mathematics*

Mathematics teacher educators have historically connected advanced mathematics to secondary teaching along two dimensions: *mathematics content at the secondary level* and *mathematical practice*. To illustrate the first dimension, *mathematical content*, we turn to specifications for advanced mathematics course content for teachers. Multiple policy documents and syllabi argue that advanced mathematics courses should connect to secondary mathematics content (e.g., CUPM, 1961; CBMS, 2001a; Murray & Star, 2013; Tucker et al., 2015).

Consider the concept of a capstone course for teachers, which is offered by many institutions in the U.S. (Cox et al., 2013), and is recommended by the guiding document *Mathematical Education of Teachers, I* (CBMS, 2001a). The various designs of capstone courses give insight into the kinds of connections that mathematics faculty see between advanced mathematics and secondary mathematics content. In their review of then-present-day capstone coursework for secondary teachers, Murray and Star (2013) found that connections took the form of generalizations or abstractions of secondary mathematical ideas (e.g., geometric transformations, factoring polynomials), or specific uses of advanced mathematics to define concepts that secondary students may encounter (e.g., $2^{\sqrt{5}}$ as an illustration of defining mathematical expressions with limits). These connections can also be viewed as interpretations of Klein's (1924/1932) notion of school mathematics from an advanced standpoint.

As for the second dimension, in the 1990s, scholars such as Cuoco et al. (1996) called attention to the importance of *mathematical practice* in teaching and learning mathematics. Recommendations for mathematics departments followed suit. As the report from CBMS (2001a) reasoned, "Teachers need to learn to ask good mathematical questions, as well as find solutions, and to look at problems from multiple points of view. Most of all, prospective teachers need to learn how to learn mathematics" (p. 8). The report theorized that, consequently, mathematics teachers must experience mathematical practices in their own learning, for then they will be more likely to foster such experiences when teaching secondary mathematics. This argument resonates with various findings from the perspective of teachers; understanding what mathematics is, and reminding them what learning mathematics feels like, were primary ways teachers found such coursework to be relevant (Baldinger, 2018; Even, 2011; Hoffman & Even, 2019).

A secondary teacher's practice benefits from robust mathematical practice and knowledge of secondary content from an advanced standpoint (Baumert et al., 2010; Sword et al., 2018). Nonetheless, teacher educators must consider: Do these two aspects alone constitute sufficient connection from advanced mathematics coursework to teachers' future work? Empirical studies establish that this potential has been unmet. Begle (1979) found that secondary students' performance was associated with neither the number of tertiary mathematics courses taken by teachers nor the average grade received by teachers in these courses. Monk (1994) also studied relationships between student performance and teacher course taking. He found that at the secondary level, there was only negligible effect after the first four mathematics courses–meaning that courses considered "advanced" had little effect. More recently, qualitative results suggest the situation has not changed. Goulding et al.'s (2003) and Zazkis and Leikin's (2010) surveys found that many teachers report their mathematical preparation is disconnected from teaching. In a study of an abstract algebra course, Ticknor (2012) found that even when secondary teachers wanted to do well in the course, and the instructor saw connections between abstract algebra and secondary mathematics, the teachers still perceived the course as irrelevant to teaching. Wasserman et al. (2018) articulated concrete reasons, given by prospective and in-service teachers, for why their knowledge of real analysis did not inform their teaching, even when they understood the material. In the next section, we provide an overview of efforts centered on a third approach to connections—*teaching practice*—which holds promise for addressing the shortcomings just discussed.

## An Emerging Program: Connections to Secondary Mathematics *Teaching Practice*

In response to both the promise of advanced mathematics to shape teachers' practice and the observed inefficacy of advanced mathematics courses to do so, secondary mathematics teacher educators have pushed for connecting advanced mathematics to teaching practice (Álvarez et al., 2020a; Artzt et al., 2011; Bremigan et al., 2011; Heid et al., 2015; Lai, 2019; Lischka et al., 2020; Wasserman et al., 2017). In this section, we characterize how various mathematics teacher educators have developed connections from advanced mathematics to *teaching practice*. We ground this characterization in a review of curricular materials and reports of enacting such materials. We show that the approaches used to connect advanced mathematics coursework to secondary teaching practice are consistent with constructs from practice-based teacher education (Forzani, 2014). In particular, the approaches resemble representations of practice, approximations of practice, and decompositions of practice (Grossman et al., 2009). Then, drawing from examples of each type of these pedagogies of practice, we raise issues for the field to consider in moving forward. In particular, we problematize how teachers may be supported–or not–to integrate their experiences in mathematics and methods courses.

**Materials Selection, Author Positionality, and Boundaries of this Review**

We sought to review curricular materials satisfying two criteria: (1) they were intended for use in advanced mathematics courses that secondary teachers may take, and (2) they sought to make explicit connections to secondary mathematics teaching practice. We also restricted this review to materials developed in the U.S.; we reasoned that AMTE is an organization whose members work primarily in the U.S., and this handbook is concerned with implications for the enactment of the AMTE standards (2017).

The authors of this chapter are also developers of such curricular materials, namely, the MODULE($S^2$) materials (Lai, Strayer, Casey, Lischka) and ULTRA materials (Wasserman, Weber, Fukawa-Connelly). We believe in the approach of connections via teaching practice, and have collectively written in various venues to advocate for this viewpoint. Our investment comes from our own review of the empirical and theoretical literature, as described in the above sections. In this chapter, we have chosen to include our own materials in our review. In doing so, we aimed to revisit our own materials relative to others and constitute part of an emerging movement.

In searching for curricular materials, we looked for both the materials themselves as well as reports of their enactment. We considered textbooks commonly used in capstone courses (as reviewed by Cox et al., 2013), those published by professional organizations of mathematicians, and the literature reporting the use of these materials and textbooks in advanced mathematics courses. As well, we conducted a Fastlane search for materials created with the support of the U.S. National Science Foundation. Table 1 displays these materials and reports. Note that in this table, "algebra" refers to the study of number systems, functions, or relations, whereas "abstract algebra" refers to the study of mathematical groups, rings, fields, or related constructs.

**Table 1**

*Reviewed materials and reports of their enactment*

| Name of authors or project | Materials reviewed | Topics addressed |
|---|---|---|
| Bremigan, Bremigan, & Lorch | Bremigan, Bremigan & Lorch (2011) | Algebra |
| Buchbinder & McCrone | Samples of materials and description of implementation as described in Buchbinder and McCrone (2018, 2020) and Buchbinder (2018) | Number Theory<br>Geometry<br>Use of conditionals |
| Capstone Mathematics | Hauk, Hsu, & Speer (2017, 2018) | Algebra,<br>Abstract Algebra,<br>Geometry |
| Enhancing Explorations in Functions for Preservice | Samples of materials and description of implementation in Álvarez et al. (2021) | Algebra |

| | | |
|---|---|---|
| Secondary Mathematics Teachers Project | | |
| Mathematical Education of Teachers as an Application of Undergraduate Mathematics (META Math) | MAA META Math (2020a, 2020b, 2020c, 2020d, 2020e, 2020f, 2020g, 2020h, 2020i) | Abstract Algebra<br>Calculus<br>Discrete Mathematics<br>Proof<br>Statistics |
| Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools (MODULE($S^2$)) | Lai & Hart (2021), Hart & Lai (2021), Aubrey et al. (2021), Casey, Ross, Maddox, & Wilson (2021a, 2021b, 2021c), Alibegović & Lischka (2021a, 2021b, 2021c), Anhalt, Cortez, & Kohler (2021a, 2021b, 2021c) | Abstract Algebra, Algebra, Geometry, Mathematical Modeling, Statistics |
| Mathematical Understanding for Secondary Teaching (MUST) | Heid, Wilson, & Blume (2015) | Algebra<br>Abstract Algebra<br>Geometry<br>Statistics<br>Proof by induction |
| Sultan & Artzt | Sultan & Artzt (2011)<br>Description of implementation in Artzt et al. (2011) | Algebra<br>Geometry<br>Statistics |
| Upgrading the Learning and Teaching of Real Analysis (ULTRA) | Materials retrieved from http://ultra.gse.rutgers.edu/. Descriptions of implementation in Wasserman and McGuffey (2021), Weber et al. (2020), Fukawa-Connelly et al. (2020), Wasserman et al. (2022), McGuffey et al. (2019), Wasserman et al. (2019). | Real analysis |
| Usiskin, Peressini, Marchisotto, & Stanley | Usiskin, Peressini, Marchisotto, & Stanley (2003)<br>Description of implementation in Winsor (2009) | Algebra<br>Geometry<br>Abstract algebra |

We now describe the approaches evinced by our review in terms of pedagogies of practice.

**Representations of Practice**

The pedagogy of practice known as *representations of practice*, which we henceforth refer to as "representations," allows novices to observe aspects of practice (Grossman et al., 2009). When teachers engage with representations, they can develop ways of noticing and understanding teaching practice. Representations can vary in comprehensiveness and authenticity, and can take a variety of forms, including short narratives of teaching scenarios, videos, animations, and case studies. Ball and Bass used representations, in the form of videos of instruction as well as written descriptions of teaching scenarios, to educate the broader mathematical community on the specialized nature of mathematical knowledge for teaching (e.g., Ball & Bass, 2003; CBMS, 2001b). The influence of their work can be found in a variety of teacher education and professional development materials today, across grades K-16.

The types of representations of practice found in the reviewed materials are shown in Table 2. Materials showed these representations of practice and then engaged teachers in evaluating, describing, or reflecting upon features of the representation of practice.

**Table 2**

*Categories of Representations of Practice*

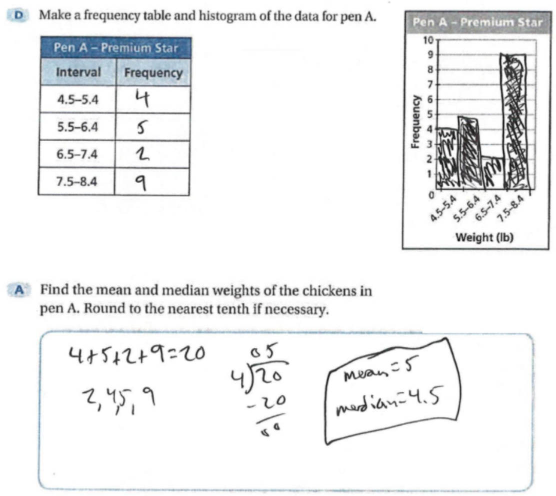
| Category | Description |
|---|---|
| Student contributions | Depictions of student talk, work, or thinking, where teachers are invisible. |
| Teacher-student interactions | Show a teacher and a student or multiple students |
| Written curriculum and assessments | Most commonly include actual or hypothetical contents of textbooks, and also include lesson plans and standardized assessment tasks |
| Personal experience with secondary level tasks | Teachers are asked to notice features of their own experiences solving a secondary level task |
| Educator's teaching of a task | An educator models teaching a task for prospective teachers, and the experience of this task itself becomes an object to notice |

Evaluating student contributions was by far the most prevalent structure for engaging with representations of practice, and evaluating teacher contributions was the second most common. A standard format for evaluating student contributions is showing teachers a sample of student work or dialogue, and then asking teachers to identify strengths and weaknesses of the mathematics shown (e.g., Bremigan et al., 2011; Casey et al., 2021a; Hauk et al., 2017, 2018; Mathematical Association of America META Math 2020a; Sultan & Artzt, 2011; Wasserman et al., 2022). Figure 1 shows two examples.

**Figure 1**

*Examples of representations of practice (Casey et al., 2021a, p. 175; Wasserman et al., 2022)*

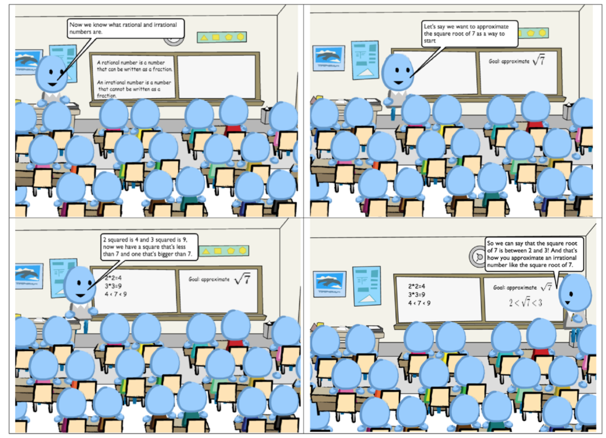
Two parts of [a student's] work on a task are shown below.




(i) Name things [that this student] has done well when analyzing the data.

(ii) Name things [that this student] needs to work towards understanding about analyzing univariate quantitative data.

Mrs. Jane, an 8th grade teacher, is teaching students about irrational numbers.






Q1. Has the teacher given a presentation that would allow students to fulfill the following CCSSM standard: "8.NS.A.2. …" ? If so, explain how. If not, why not?

The stance of evaluating raises the issue of how teachers frame discourse in terms of personal knowledge. In most materials reviewed, the prospective teachers were positioned to evaluate student and teacher talk with advanced mathematics content learned in the course. But teachers do not only view classroom discourse through a mathematical lens; they also consider implications for students' affect, beliefs, and relative position. Ideally, teachers are able to fluidly transfer lenses and integrate inferences from their observations. In one moment, they may consider mathematical implications. The next moment, they may wonder about consequences for students' perception of their agency. Later, they may reflect on these evaluations. We hypothesize that in methods courses, teachers may be asked to use a pedagogical lens to evaluate discourse, and also a mathematical lens that does not include advanced mathematics knowledge. If this hypothesis is true, then there is a disconnect between how teachers learn to evaluate representations of practice in methods courses and in advanced mathematics courses.

Whether teachers are asked to evaluate, describe, or reflect upon representations of practice, there is also a question of authenticity to actual secondary mathematics teaching practice, and how uses of representations influence teachers' developing notions of teaching. Notice that we problematize authenticity–rather than, say, dismissing inauthenticity out of hand. Positive effects are reported both by projects that sought to create representations typifying recurrent classroom interactions (Strayer et al., 2021; Wasserman & McGuffey, 2021) and those where the materials featured representations that may not be as typical (Sultan & Artzt, 2010; Winsor; 2009), such as, “A student wants to know if the expression $(\sqrt{2})$^$(\sqrt{2})$^$(\sqrt{2})$ is rational or irrational. What would you say?” (Sultan & Artzt, 2010, p. 252). There is not empirical evidence at this point to say that one stance is unconditionally better than the other.

In view of the reviewed sources, we hypothesize that implementation is critical to how and whether teachers draw connections to teaching practice. In Artzt et al.’s (2011) and Winsor’s (2009) studies, prospective teachers taught their peers, and hence they interacted with their peers-as-learners. In the projects reported by Strayer et al. (2021) and Wasserman and McGuffey (2021), teachers were given time to reflect on representations and produce a reply to typical secondary students’ thinking, but there is no actual interaction between teachers and other persons. More examination is needed to identify how teachers’ experiences are mediated by implementation.

### Approximations of Practice

The pedagogy of *approximations* of practice allow novices to simulate practice so they can attend to particular aspects of practice, rather than all aspects of the complex, relational practice that is teaching (Grossman, 2009). Approximations may vary in authenticity and complexity (see Table 3).

**Table 3**

*Categories of Approximations of Practice*

| Approximation Category | Description |
|---|---|
| Responding to students’ mathematical contributions | Attend to features of students’ contributions, interpret them, and then determine how to respond |
| Evaluating students’ work | Determine or quantify mathematical validity of students’ work |
| Explaining content | Produce a written or spoken account of how one might explain a mathematical idea such as a procedure, concept, or connection |
| Teaching a full lesson | Either teaching a lesson of the advanced mathematics course in which teachers were enrolled, or teaching a lesson to secondary students |

Across the multiple secondary education projects reviewed, the majority of approximations featured the practices of responding to students’ mathematical contributions, evaluating students’ work, and explaining content (e.g., Álvarez et al., 2020b; Alibegović & Lischka, 2021b; Aubrey et al., 2021; Bremigan et al., 2011; Buchbinder & McCrone, 2020; Hauk et al., 2017; MAA META Math, 2020b; Sultan and Artzt, 2011; Wasserman et al., 2022). For instance, the Capstone Math materials provided teachers with a set of student work samples and asked teachers to work in groups to “sort the work into categories that represent different ways of thinking and/or difficulties” (Hauk et al., 2017, p. 9). Throughout the MODULE($S^2$) materials, teachers are asked to consider samples of student work in response to a task, and then to either write a narrative describing how they would “conduct a whole class discussion which will allow you to elicit student thinking, … with specific use of the [students’] work,” or to record a video of themselves “providing a response to [the students]” that highlights the student’s thinking as well as engages the student in the intended mathematics (see Figure 2).

**Figure 2**

*Example of approximation of practice (Alibegović & Lischka, 2021b)*

As students are working independently on rotations of a flag around a point, you observe two students with the following work completed.


Record a video of yourself providing a response to both Student 1 and Student 2 where you include a summary of what each student might be thinking and what is worthwhile or reasonable about that student’s thinking, and a response to each student that does one or more of the following: helps the student finish their thought, prompts the student to investigate an error, or helps the student move forward in their thinking.

These examples constitute approximations of practice because they simulate recurrent work in teaching. In these examples, this work includes diagnosing student conceptions, orchestrating discussion, and posing questions to students. They are also approximations in that they are artificial: the teachers have far more time and latitude to come up with and revise their ideas than they would have in front of actual students. At the same time, these examples raise the issue of diversifying the teaching practices being approximated. Instruction is more than explaining content and responding to student work. It also includes setting up norms and routines for disciplinary work, setting learning goals, and adjusting goals and instruction, among other practices.

One way to weave in more teaching practices is to teach full lessons. We found two types of this pedagogy. First, as reported by Artzt et al. (2011) and Winsor (2009), teachers might teach a session of the capstone course to their peers. In both cases, the teaching experience imparted a deeper appreciation of the depth of content knowledge entailed in teaching a topic. Further, teachers believed that this lesson learned would apply to their own future secondary teaching. In Buchbinder and McCrone’s (2020) case, the teachers planned and taught actual secondary class lessons featuring proving and reasoning. The teachers in Buchbinder and McCrone’s study also developed an appreciation of the content knowledge demands of teaching. Further, they replicated models of tasks that they had experienced in their own advanced mathematics course.

In all of these cases, teaching full lessons constituted an approximation of practice: the teachers received far more support and planning time than a practicing teacher would.

Within the materials reviewed, the scope of approximations of practice was either relatively small, as exemplified by the approximation shown in Figure 2, or relatively large, as in teaching a full lesson. Moreover, the range of teaching practices was narrow. To diversify the teaching practices that are approximated, it may be necessary to design new types of approximations of practice. Such new approximations may at once leverage advanced mathematics and also give teachers the opportunity to approximate practices such as setting up discourse routines or adjusting goals and instruction. Such designs remain an open problem for mathematics teacher education.

### Decompositions of Mathematical Practice with Implications for Teaching Practice

The pedagogy of *decompositions* of practice supports educators by providing maps of teaching practice that can be used to attend to prospective teachers' enactment of practice (Grossman et al., 2009). Decompositions of practice in the reviewed materials were rare. However, there were several decompositions of *mathematical* practice rather than teaching practice across the materials. These included decompositions of the ways that secondary students may understand congruence, similarity, or dilation (Hauk et al., 2018) or decomposing the sufficiency or insufficiency of considering examples in proving or disproving certain conditional statements (Buchbinder & McCrone, 2020). These decompositions of mathematical practice have implications for teaching practice and may serve as resources for both advanced mathematics and methods instructors in articulating mathematical practice in teaching.

## Reflections on the Current State and Pushing the Field Forward

We began this chapter by describing historical approaches to connect advanced mathematics and secondary mathematics teaching. Yet, despite the rationales underlying these historical approaches, they largely were ineffective: many teachers viewed advanced mathematics courses as irrelevant to teaching. A practice-based approach to making advanced mathematics courses work for teacher education takes into account this lesson: educators must go further than connecting advanced mathematics to what teachers need to *know*; they must connect advanced mathematics to what teachers actually *do*.

### Promising Evidence for a Practice-Based Approach

We see two points of evidence in support of adopting pedagogies of practice as a way to connect advanced mathematics and secondary mathematics teaching practice. First, mathematics faculty have been willing to adopt materials with this approach. Some of the reviewed materials are also among the most common textbooks for capstone courses (Cox et al., 2013); Álvarez et al. (2022) reported that instructors using the META Math materials used the tasks featuring teaching scenarios; additionally, as authors of some of the reviewed materials, we know firsthand of faculty from across the US using the materials, and for multiple years.

Second, in contrast to earlier studies (e.g., Goulding et al., 2003; Zazkis & Leikin, 2010), pilot studies of practice-based approaches to advanced mathematics courses suggest that teachers are able to describe connections between advanced mathematics and their future teaching (cf., Fukawa-Connelly et al., 2020; McGuffey et al., 2019; Wasserman & Galarza, 2018). Artzt et al. (2011) and Winsor (2009) both reported that the teachers in their courses developed an appreciation for how mathematical proofs and underlying reasoning could inform their diagnosis of students' thinking as well as their responses to students' questions. Moreover, when teachers are teaching secondary students, they may incorporate ideas from their advanced mathematics courses. Wasserman and McGuffey (2021) of the ULTRA project observed former students of their project teaching in their own secondary classrooms. They found that teachers attributed some teaching moves to their experiences in real analysis as designed by the ULTRA project. Buchbinder and McCrone (2020) found that prospective teachers used task designs they had encountered in their advanced mathematics courses when they were asked to design and teach a proof-based lesson for secondary students while student teaching. In open-ended responses to MODULE($S^2$) surveys, multiple teachers who took advanced mathematics courses using these materials pointed to the usefulness of the approximations of practice.

**Pushing the Field Forward: Essential Questions for the Future of Advanced Mathematics Courses for Teachers**

Looking forward while holding a view across the pedagogies of practices used, we call attention to five areas to move the field forward in connecting advanced mathematics to secondary mathematics teaching practice in meaningful ways.

(1) *Methodological and theoretical considerations for how knowledge of advanced mathematics–especially mathematical practice–shapes secondary teaching practice.* Across all reviewed projects, we saw connections via both mathematical content and mathematical practice to the work of teaching secondary mathematics. The prominent connection across all the materials was via mathematical content. In such instances, it was possible because the advanced mathematical content gave a framework for viewing the secondary content as a special case. That is, studying advanced mathematics helped to situate the secondary content in a mathematically meaningful way. By contrast, when the materials made connections through mathematical practice, the connection was by way of analogy, in drawing on how processes engaged in doing mathematics could be productive for teaching mathematics as well. Notably, these instances appeared primarily in more recent materials, suggesting that the opportunity presented by, and the need to be explicit about, these mathematical practices in advanced coursework–and their relationship to teaching practice–is something that has become more evident over time.

The inclusion of practice connections raises questions about how to best discuss mathematical practices in these courses. Mathematics educators have identified various similarities between advanced and secondary mathematics, and how these similarities may inform teaching. But mathematical practice evolves across K-20 years (Stylianides et al., 2009; Tall, 1991). Mathematics teacher educators thus far appear to leverage similarities, but not differences, in mathematical practice. Yet, an advanced mathematics course must aim to engage teachers in advanced mathematical practice.

We ask: How can differences between advanced mathematics practice and secondary mathematics practice be a resource rather than a limitation to meaningful connections between advanced mathematics and teaching practice?

(2) *Diversifying depictions of practice and issues of inclusion.* As noted earlier, there is a narrow band of teaching practices featured in approximations of practice in advanced mathematics courses. A predominant one was explaining content, which also mirrors the way many situations in teaching were represented. This limitation in the way pedagogies of practice were accomplished in the reviewed materials may reinforce the (undesirable) notion that teaching equates to explaining. Thus, we ask: How can this practice-based approach be diversified to include additional teaching practices? How can the diversity of teaching–beyond just explaining content–be captured and incorporated into advanced mathematics coursework? In exploring such questions, it is also essential to identify ways that do not overload advanced mathematics instructors nor expect too much from teacher candidates who are novices. Only one reviewed project leveraged clinical experiences (Buchbinder & McCrone, 2020). Exploring further whether there might be clinical experiences that meaningfully leverage knowledge of advanced mathematics to, for instance, support students in constructing sound geometric proof, or connect algebra and geometry.

Furthermore, with only two exceptions, the reviewed projects and reports did not appear to intentionally take on issues of social justice and equity in their use of pedagogies of practice. Indeed, the broader literature on teacher education suggests that equity and social justice may be a blind spot in practice-based teacher education more generally (Kavanagh & Danielson, 2014). We ask: Are there ways to take on issues of social justice and equity in advanced mathematics courses, given the complexity of the issue? As a smaller point, but one connected to these broader issues, we note that rarely did materials prompt teachers to identify mathematical strengths of student work without it being paired with identifying weaknesses of student work. Doing so may promote a deficit-based perspective about students as mathematical thinkers, rather than an asset-based perspective. Framing pedagogies of practice may be one realm where issues about equity and inclusion can be taken up.

(3) *Identifying explicit ways to acknowledge and discuss pedagogical aspects of mathematics teaching practice*. Multiple projects featured an explicit way to characterize and discuss pedagogical (and not just mathematical) aspects of teaching. The specific details differed widely across these three projects, but the key point is that in each, the discussion of pedagogy had some explicit operationalization. In META Math, a collection of five different types of connections guided the design of the materials, and prescribed aspects of what teachers should be able to do with their mathematical knowledge in their teaching (Álvarez et al., 2020a). Sultan and Artzt (2011) aimed for teachers to create lessons that aligned with ten dimensions of mathematics teaching competence. In MODULE($S^2$), responses to teaching situations were considered with respect to Rowland and colleagues' (2013, 2016) Knowledge Quartet and Simon's (2006) notion of mathematical understanding (Lai et al., 2021). When asking teachers to engage in a specific pedagogy of practice, even while primarily discussing mathematics, these projects drew on these

conceptions to structure pedagogical conversations and design pedagogies of practice. The MUST project designed situations around the mathematical context of teaching, and specified some of the components of this aspect of teaching (Heid et al., 2015). In ULTRA, a set of eight written teaching principles explicitly guided any conversations about teaching; the principles were described in relation to both mathematical and pedagogical teaching practice (Wasserman et al., 2022).

We ask: What can we learn about how discussions of pedagogy can be incorporated into advanced mathematics courses from projects that did so? How might discussions of pedagogy in mathematics courses differ from discussions of pedagogy in methods courses? What do potential differences mean for integrating teachers' experiences in advanced mathematics and methods courses?

(4) *Differentiating and coordinating how pedagogies of practice are taken up in mathematics content and methods courses.* Initial studies indicate the promise of connecting advanced mathematics to teaching practice via pedagogies of practice. Yet we recognize that pedagogies of practice, as taken up in mathematics courses, may necessarily differ from pedagogies of practice as taken up in methods courses. The pedagogies of practice reviewed here do not always resemble the pedagogies of practice used in methods courses. Where there is overlap, the content and practices are more specific to the intended advanced mathematics than would likely be possible in a methods course. On the one hand, it is reasonable that the pedagogies of practice have qualitatively different characteristics: methods courses and advanced mathematics courses have different objectives, and different objectives will lead to different task design. On the other hand, these differences raise a question of how prospective teachers, who will encounter these different pedagogies of practice, will integrate rather than silo their teacher preparation experiences. We ask: How can pedagogies of practice developed in mathematics content and methods courses be meaningfully coordinated? What funding and support structures might be needed to foster the collaboration amongst mathematicians, mathematics educators, teachers, and teacher educators that is necessary to do this interdisciplinary work?

In raising these questions, we call for mathematics teacher educators and other stakeholders to learn from current research and practice so as to innovate in critical new directions to improve the enactment of the AMTE Standards for Preparing Teachers of Mathematics (2017) for secondary teacher education.

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[i] These indicators state that undergraduate mathematics courses should attend to mathematics relevant to teaching (P.2.1) and build mathematical practices (P.2.2); and methods courses should instill mathematics as deep and meaningful (P.3.1) and offer practice-based experiences (P.3.4).

[ii] As a case in point, recognizing that the set of invertible functions under composition form a group provides secondary teachers with a deeper, richer, and more connected notion of inverse functions. And, given the application of inverse functions in solving equations in school mathematics, it is reasonable to think that teachers would benefit from this strengthened knowledge base.