

Characterizing Prospective Secondary Teachers'  
Foundation and Contingency Knowledge for Definitions of Transformations

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Siloing subject matter risks students who then silo ideas. When the students are prospective mathematics secondary teachers, and their mathematics and pedagogy courses have little overlap, the students may believe that knowledge from one course is contextually inappropriate for the other course. Indeed, despite teachers' many opportunities to learn tertiary mathematics (Tatto & Bankov, 2018; Hill, 2011) there has historically been little evidence that tertiary mathematics course-taking influences teachers' pedagogy (Zazkis & Leikin, 2010). Furthermore, multiple studies have documented that many secondary teachers themselves do not find tertiary mathematics courses relevant to their careers, (Goulding et al., 2003; Wasserman et al., 2018; Zazkis & Leikin, 2010), even when they have done well in the mathematics course (Wasserman & Ham, 2013). Though researchers have identified instances where tertiary mathematics can shape individual secondary mathematics teachers' decisions (e.g., Baldinger, 2018; Zazkis & Mamolo, 2011), it was not until recently that a study recorded the direct influence of a tertiary mathematics course on future teachers' instruction (Wasserman & McGuffey, 2021). This finding was due to intentional connections between particular university mathematics (real analysis) and secondary mathematics teaching.

Our work, which involves data from the *Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools* (MODULES) Project, is guided by the challenge and desire for meaningful integration of learning mathematics and learning to teach mathematics (Baumert et al., 2010). Mathematics encountered by prospective secondary teachers may range from geometry to algebra, to statistics, to mathematical modeling (Tatto & Bankov, 2018).

In this paper, we review a promising type of task, which we refer to as *prompts for teacher-created representations of practice* (cf. Grossman, 2009). These prompts can be used across content areas, and engages prospective secondary mathematics teachers in creating their own images of teaching practice in response to a given teaching scenario. We use the term *representation* because the teachers are representing their image of teaching in responding to these tasks. We then describe a framework for describing the potential range of teachers' mathematical knowledge and skill in using mathematics in teaching. This framework attends to teachers' *foundational content knowledge* and *actions contingent upon specific student thinking* (cf. Rowland, 2013; Rowland et al., 2016). The former includes teachers' personal understanding of mathematics underlying the concepts addressed in the teaching scenario. The latter incorporates how teachers frame explanations and questions in terms of student thinking. As a preliminary 'proof of concept' of analysis with this framework, we examined teachers' responses to four tasks in one content area: geometry from a transformation approach. Finally, we consider how our work might generalize to other mathematical domains.

The following questions guided this study: *What foundational knowledge and contingent actions are observable in teacher-created representations of practice? How can we characterize quality of foundational knowledge and contingent actions?*

## 1. Background

Secondary teachers take a range of university mathematics, from first year courses in calculus to courses such as real analysis, which are also required of those pursuing graduate studies in mathematics (Ferrini-Mundy & Findell, 2001; Tatto & Bankov, 2018). Programs hope that by requiring these courses, secondary teachers will teach mathematics with greater perspective and accuracy (Conference Board of the Mathematical Sciences [CBMS], 2012; Ferrini-Mundy & Findell, 2001; Winsløw & Gronbæk, 2014). However, Monk (1994), who studied relationships between secondary student performance and teacher course taking, found that there was only a negligible effect after the first four mathematics courses. In other words, any course taken after the first or second year, which would likely include courses specifically designed for secondary teachers, or any advanced courses, had little effect.

### 1.1 Perceived discontinuity between university mathematics and secondary teaching

Moreover, as has been well-documented, many secondary teachers find their tertiary mathematics education irrelevant to their teaching. The studies of Goulding et al. (2003), Ticknor (2012), and Zazkis and Leikin (2010), among others, found that to these teachers, the university curriculum appeared to focus on topics unrelated to the K-12 curriculum. Even among teachers who found their mathematical preparation useful, they are rarely able to cite specific instances of how tertiary mathematics influenced their teaching (Wasserman & Ham, 2013; Zazkis & Leikin, 2010). Some prospective teachers may view the utility of university mathematics content using a “transport model”, focusing their attention on how well the exact mathematical explanations from their university mathematics courses will transfer to teaching secondary mathematics (Wasserman et al., 2018, p. 83).

## 1.2 Connecting university mathematics to secondary teaching via applications of mathematics to teaching

Historically, attempts to connect tertiary mathematics to secondary mathematics teaching tended to focus almost exclusively on the *mathematics* (e.g., The Panel on Teacher Training, 1971, 1983; CBMS, 2001; Kerr & Lester, 1982; Mathematical Association of America, 1983), with little attention to mathematics *teaching practice*.

We, along with multiple others, take the stance that addressing the discontinuity problem requires connecting tertiary mathematics and secondary mathematics teaching practice (Álvarez et al., 2020a; Heid et al., 2015; Lischka et al., 2020; Ticknor, 2012; Wasserman et al., 2018; for a review, see Lai et al., in press). In our work, we take the approach of enhancing curricula for tertiary mathematics courses. To so do, we incorporate activities that *apply mathematics to teaching* (or that are *application problems*) (Álvarez et al., 2020a; Stylianides & Stylianides, 2010). We define such activities as tasks where prospective teachers consider a secondary teaching situation to which the university mathematics can be leveraged, and then draw on this mathematics to respond to the situation in ways appropriate for secondary teaching.

As Álvarez et al. (2020a) observed,

Including applications to teaching in mathematics content courses ... can advance content learning goals and meet the needs of prospective secondary mathematics teachers as they make connections between the advanced mathematics they are learning, the mathematics they will teach, and the complex human context that is central in the work of teaching (p. 17).

Furthermore, such activities may help secondary teachers “transcend the transport model” (Wasserman et al., 2018, p. 87). The tasks allow prospective teachers to use university mathematics understandings during the work of secondary teaching.

### 1.3 Teacher-created representations of practice: An application of mathematics to teaching

Applying mathematics to teaching may come in different forms: analyzing students' mathematical reasoning, explaining a mathematics teacher's decision, or posing questions in response to student work. Some tasks in recent projects ask prospective secondary teachers to describe, or even write a script for, what they would say or do to explain particular secondary-level concepts or upon reviewing secondary student work (e.g., Álvarez et al., 2020a, Lischka et al., 2020, Wasserman et al., 2018). We say that the prospective secondary teachers' responses to these tasks are *teacher-created representations of practice*, because the teachers are creating a representation of their image of secondary mathematics teaching practice (cf. Grossman, 2009). We use the phrase *prompt for teacher-created representations of practice* to refer to the tasks that elicit teacher-created representations of practice.

For instance, a prompt may present secondary students' explanations for an algebraic solution, and then ask teachers to pose questions to students to help uncover arithmetic assumptions used. Or, a prompt may present secondary students' attempts to construct a graphical display to represent the association between two quantities, and ask teachers to comment as they would to a student about the quality of the work. The teaching practices involved here—such as posing questions and interpreting student work—require mathematics to carry out well (e.g., Ball, Thames, & Phelps, 2008; Baumert et al., 2010).

Teachers' responses to such prompts can therefore showcase their recognition of mathematics relevant to the teaching scenario, as well as how they use this mathematics. Monk's (1994) results suggest that if there is a relationship between course taking and teaching, it may be nuanced. We explore the hypothesis that explicit connections to teaching are part of this nuance.

#### **1.4 The promise of connecting university mathematics to secondary teaching practice**

The potential benefits of connecting mathematics to teaching practice are both practical and methodological. First, practicing teachers may attribute their teaching moves to their experience with teacher-created representations of practice featured in university mathematics courses, as Wasserman and McGuffey (2021) found when observing and interviewing former students of their project. Moreover, in another study using our project's data, teachers attributed increased confidence in their teaching practice to creating representations of practice (Lai et al., 2023).

Methodologically, teacher-created representations of practice may be a way to elicit and evaluate mathematical knowledge for teaching – the mathematical ideas, concepts, skills, and sensibilities entailed and manifested the recurrent work of teaching (Ball et al., 2008). Early in the scholarship of mathematical knowledge for teaching, Ball and Bass (2003) proposed that a way to assess knowledge of mathematics relevant to teaching was to pose questions where teachers respond to a given mathematics teaching scenario. Since then, multiple projects at the secondary level seeking to assess mathematical knowledge for teaching have used this principle as well (e.g., Baumert et al., 2010; Herbst & Kosko, 2012; McCrory et al. 2012; Mohr-Schroeder et al., 2017). Teacher-created representations of practice follow this lineage, with an instantiation in curriculum rather than assessment instruments.

As data, teacher-created representations of practice may bring more nuance to describing the quality of mathematical knowledge used and how it is applied to teaching. Álvarez et al. (2020b) used teacher-created representations of practice to analyze teachers' propensity to validate student thinking, the extent to which they addressed visual representations of function from the student's perspective, and a tendency to look for rules. Weber et al. (2020) used teacher-created representations of practice to articulate how teachers may use known procedures when analyzing

student work, and how well teachers connected different criteria for a mathematical definition. Álvarez et al.(2020a) identified a tendency of university students to focus on computation rather than underlying concepts in sample secondary student work, a finding that led them to re-design some of their prompts for teacher-created representations of practice.

One of the greatest needs in secondary teacher education is designing ways for teachers to learn both mathematics and how to apply that mathematics to teaching. The mathematics known by teachers presents a ceiling for how well they can communicate mathematics with their students (Baumert et al., 2010). Further, as Lai et al. (in press) argued that there is a need to identify explicit ways to characterize the mathematical and pedagogical qualities of applications of mathematics to teaching. In this way, mathematics teacher educators can understand better how mathematical practice can shape secondary teaching practice. This study contributes to this need. We characterize the quality of the mathematics used in teacher-created representations of practice, as well as the quality of how sample secondary students' work is taken up by teachers.

## **2. Conceptual Perspective**

### **2.1 Mathematical knowledge in and for teaching (MKT)**

In a presidential address to the American Educational Research Association, Shulman (1986) called for attention to a “missing paradigm” problem in the study of teaching (p. 6): research that focused simultaneously on content and pedagogy. Multiple research groups responded to Shulman’s address, resulting in different approaches to conceptualizing knowledge use in teaching. These include:

- Ball and colleagues’ “mathematical knowledge for teaching” that elaborates Shulman’s notions of content knowledge and pedagogical content knowledge (e.g., Ball & Bass, 2003; Ball et al., 2008), and which was re-conceptualized at the

secondary level by Baumert et al. (2010);

- Thompson and Thompson’s (1996) inquiry into “mathematical knowledge for teaching” and “knowledge for conceptual teaching”;
- Davis and colleagues’ scholarship on “mathematics-for-teaching” (e.g., Davis & Simmt, 2006); “mathematical knowledge in teaching” as discussed in a seminar led by Ruthven and Rowland (2007);
- Heid and collaborators’ development of the notion of “mathematical understandings for secondary teaching” (Heid et al., 2015); and
- Rowland and colleagues’ (2013, 2016) “Knowledge Quartet”, which identifies and categorizes teaching routines and moments where knowledge use may be observed.

This is but a sample of scholarship on mathematical knowledge that informs teaching, and illustrates the variety of terminology in the literature.

For our project, the most salient aspect of this wide scholarship is its common thread: that teachers use content knowledge in recognizing, understanding, and responding to mathematical situations, considerations, and challenges that arise in the course of teaching mathematics. We refer to this knowledge as mathematical knowledge in and for teaching (MKT).

Synthesizing ideas from these frameworks shaped our view of how mathematical knowledge influences teaching practice, and informed the design of prompts for teacher-created representations of practice. First, Baumert et al. (2010) re-conceptualized Ball and colleagues’ content knowledge to apply to secondary level as “a profound mathematical understanding of the curricular content to be taught” (p. 142). We have adopted a similar point of view. Second, Baumert et al. (2010) distinguished content knowledge from pedagogical content knowledge, the knowledge needed for making mathematics comprehensible to students (Shulman, 1986). These



distinctions echo those made by other scholars (e.g., Ball et al., 2008; Heid et al., 2015; Rowland, 2013; Shulman, 1986). Further, Baumert et al. (2010) argued that although pedagogical content knowledge appears to have a stronger role on secondary student outcomes than content knowledge does, mathematics teacher educators must attend to content knowledge. One of Baumert et al.'s results is that a teacher's content knowledge determines how much pedagogical content knowledge the teacher will be able to learn.

One critique of Ball and colleagues' conception of knowledge for teaching is that they do not explicitly discuss its dynamic aspects. That is, teachers' knowledge is not static, but rather develops through, and perhaps even manifests itself in practice (Davis & Simmt, 2006; Ruthven & Rowland, 2007). This view supports our stance that teachers' university mathematics experiences would be enhanced by incorporating connections to teaching practice. Teacher-created representations of practice may allow mathematical knowledge to manifest.

In designing prompts for teacher-created representations of practice, we took into account the importance of supporting students through adaptive instruction (Baumert et al., 2010; Thompson & Thompson, 1996) and of cultivating mathematical practices such as defining, justifying, sense-making, or representing (Heid et al., 2015). All of our prompts for teacher-created representations of practice feature a student-level task, sample student work on that task, and a goal for teaching that involves engaging students in a mathematical practice.

Finally, in analyzing teacher-created representations of practice, we lean most on Rowland and colleagues' work on the Knowledge Quartet, whose details we will soon discuss. For now, we remark that the reader may wonder why this framework and not one of the many others. Our response is that we chose it for methodological reasons. We wanted a framework that articulated where exactly mathematical knowledge manifests in teaching practice, so that we would be able

to identify such locations in teacher-created representations of practice. Rowland (2013) observed that this feature distinguishes the Knowledge Quartet from other frameworks for mathematical knowledge in and for teaching.

## 2.2 The Knowledge Quartet and its dimensions

Soon after researchers began to conceptualize mathematical knowledge in and for teaching, multiple projects sought to link this knowledge to educational outcomes. At elementary and secondary levels, projects used assessments designed to measure teachers' mathematical knowledge and largely found associations between teachers' mathematical knowledge, learning gains, and desirable teaching qualities (e.g., Baumert et al., 2010; Hill et al., 2005; Hill et al., 2008; Rowland et al., 2000).

Rowland et al.'s (2000) study, of 150 London-based primary teachers, motivated the following puzzle: "If superior content knowledge really does make a difference when [teaching mathematics], it ought somehow to be observable in the *practice* of the knowledgeable teacher." (p. 17; emphasis in the original). They wanted to know where knowledge could be observable so as to "frame a coherent, content-focused discussion" (Rowland, 2013, p. 21) between a teacher and someone who was observing the teacher for the purpose of giving feedback to the teacher about their teaching. Rowland and his colleagues then undertook an observation and video study using a purposive sample of 12 participants drawn from a pool of 149 primary teachers to reflect a range of outcomes based on performance on an assessment of their mathematical knowledge. Using a grounded theory approach (Glaser & Strauss, 1967; as cited in Rowland, 2013), they generated four broad categories of codes for teachers' knowledge and knowledge use in teaching.

**Knowledge Quartet for use at secondary level.** Since the conclusion of Rowland's initial studies, researchers have validated these categories as describing knowledge and its use beyond

primary level and beyond the UK. Weston et al. (2012) reported on a cross-national study to determine whether the Knowledge Quartet would be a feasible framework for analyzing secondary level data. Although they identified more codes for moments of teaching where mathematical knowledge is observable, their four broad categories from the primary level study remained stable across a team of 15 researchers from 7 countries working with 55 episodes of secondary teaching. These categories make the four dimensions of the Knowledge Quartet.

**Dimensions of the Knowledge Quartet and where to observe them.** Two dimensions are *Foundation* (knowledge and understanding of mathematical ideas, the nature of mathematics, as well as principles of mathematical pedagogy) and *Contingency* (the ability to respond to unanticipated events ranging from network outages to learners' alternative strategies). The moments of teaching where Foundation knowledge is observable include “awareness of purpose”, “identifying errors”, “overt display of subject knowledge”, and “use of mathematical terminology” (Rowland et al., 2016, p. 1). The moments of teaching where Contingency knowledge is observable include “deviation from agenda”, “responding to students' ideas”, “use of opportunities” (p. 1). The remaining dimensions are *Transformation* (presenting ideas to learners) and *Connection* (cohering ideas over time); these are observable in moments such as “choice of examples” or “decisions about sequencing”, respectively (p. 1).

We highlight Foundation and Contingency because we focus most on them. We now discuss our view of teacher-created representations of practice, how we operationalized Foundation and Contingency, and why our analysis used only these dimensions rather than all four.

### 2.3 Teacher-created representations of practice

As Amador et al. (2017) noted, there is power in having prospective teachers “take on an active role as *designers* (rather than *viewers*) of classroom scenes” (p. 160; emphasis in the

original). We view teacher-created representations of practice as the result of teachers' design work with samples of secondary student work and a statement of an intended goal (e.g., "to advance students' understanding of how a definition connects to a procedure").

Teacher-created representations of practice can give a window into teachers' conceptions of mathematics teaching as well as their use of mathematics in teaching. We posit that when teachers share an image of envisioned teaching using sample student work, they – like teachers with classroom artifacts – draw on their knowledge and commitments (Brown, 2009).

## **2.4 Characterizing Foundation and Contingency in teacher-created representations of practice**

Our study and Rowland and colleagues' studies have key similarities and also critical differences. Rowland and colleagues used videos of teaching across multiple topics in multiple schools. We examined teacher-created representations of practice responding to a limited set of prompts. Hence, we found it useful to delimit and elaborate the dimensions we used: Foundation and Contingency. We now discuss our operationalizations and then say why our analysis did not consider the remaining dimensions.

**Foundation.** We delimited the Foundation dimension to knowledge of mathematics, because of our interest in content coursework. Second, the dependence of Foundation on *mathematical understanding* suggested that we be theoretically clear about a conception of mathematical understanding. We used Simon's (2006) characterization: mathematical understanding is the "learned anticipation of the logical necessity of a particular pattern or relationship" (p. 364). For instance, we consider understanding mathematical procedures to include relating that procedure to its underlying definitions or concepts, and to anticipate doing so when explaining procedures or troubleshooting a use of a procedure.

**Contingency.** We delimited Contingency to the ability to use *given* student thinking in teacher-created representations of practice. By *given*, we mean the student contributions explicitly provided in the prompt for teacher-created representations of practice rather than, say, imagined by a prospective teacher. We constrained Contingency in this way because all our prompts provided sample secondary student thinking, and we posited that integrating *given* student thinking approximated for a prospective teacher an encounter with potentially unexpected student contributions.

**Documenting the potential range of Foundation and Contingency knowledge.** Rowland and colleagues' work results in a framework for identifying instances where mathematical knowledge may be used in teaching, but it does not result in a framework for characterizing variation of such use. Weston (2013) used the Knowledge Quartet dimensions to quantify mathematical knowledge in teaching for the purpose of informing programs of teacher education. As she noted, she sought to use “consistent observation-based data across multiple trainees and multiple lessons in order to inform ITE programmes, *rather than to support individual teacher development*” (p. 289; italics ours). Our purpose is to characterize the potential range of knowledge use in dimensions of the Knowledge Quartet apparent in individual teachers' work, in ways that may support teachers' development.

## **2.5 The choice to use only Foundation and Contingency rather than all four dimensions**

We focused on Foundation and Contingency in our work because they are the most visible in the teacher-created representations of practice we analyzed. In other words, the evidence from the teacher-created representations of practice was available and appropriate. Our results reported in this article also suggests that the evidence is sufficient for the purpose of supporting teachers' development. Teacher-created representations of practice provide a snapshot of

teacher's use of knowledge and responding to sample student work. The scope of the prompts is rich enough for at least some proportion of teachers to demonstrate mathematical understanding in the sense of Simon (2006). The prompts are also expansive enough for at least some proportion of teachers to explicitly address the student work. We note that while the argument of available and appropriate may be supported by the design of tasks, an argument of sufficiency can only be made *post hoc*, rather than *a priori*. For this reason, we point to our present work as part of the argument.

We originally set out to use all four dimensions, and found that whereas we had robust evidence for Foundation and Contingency, our evidence for Transformation and Connection seemed thin. In particular, Foundation is identified through teachers' use of mathematical terminology, their awareness of their purpose, and their command of the mathematics. Contingency can be examined through teachers' responses to student ideas. We were able to identify these moments consistently across teacher-created representations of practice. However, Transformation relies on teachers' choice of examples of instructional materials, but these were highly constrained by our prompts for teacher-created representations of practice. Connection involves ideas over time, and teacher-created representations of practice often focused on a relatively small chunk of teaching.

Nonetheless, Foundation and Contingency, as we use them, do echo the distinction between content knowledge and pedagogical content knowledge, while also benefiting from Rowland's work to articulate where to identify these areas. Ultimately, we do believe it would be possible to detect Transformation and Connection, and potentially ranges of these dimensions, in teacher-created representations of practice. However, the prompts for these representations would have to be designed differently than the ones that we used.

### 3. Context

This study was conducted as part of the MODULES Project (n.d.), which seeks to connect university mathematics coursework in secondary teacher preparation programs to secondary teaching practice. MODULES materials address the areas of algebra, geometry, mathematical modeling, and statistics. Throughout all MODULES materials, there are opportunities for teachers to apply mathematics to teaching. Example activities include considering student thinking, discussing common student conceptions, and connecting to learning standards. These opportunities arise at least once per every two intended weeks of curriculum material use.

Each set of materials was written by an authorship team composed of mathematicians, mathematics educators, and practicing secondary mathematics teachers. Teams ranged from 3 to 5 persons each. The second author of this article co-wrote geometry materials, and the first author co-wrote algebra materials. The authorship teams collectively developed common writing standards regarding prompts for representations of practice and approach to content.

For materials written after the adoption of the writing standards for [BLINDED] project, prompts for teacher-created representations of practice describe a teaching scenario that specified a student-level task, a goal for the scenario involving engaging students in a mathematical practice, and images of secondary student work created by secondary students or a portion of a classroom discussion that came up while working on this geometry problem. Classroom teachers on authorship teams collected samples of actual student work to represent in applications of mathematics to teaching.

The project advertised the materials to listservs and professional networks of mathematics and mathematics education university faculty. Instructors from seven universities agreed to participate in this study. They received in-person training the summer prior to teaching with the

materials as well as ongoing support for using the materials during the semester via online video calls. Instructors reported that these sessions, along with the materials themselves, supported successful implementation. Materials were piloted by members of the authorship teams and improved through iterative cycles throughout the project grant period in part informed by instructor feedback and project goals (Bryk et al., 2015).

### **3.1 Mathematical Context for Teacher-Created Representations of Practice**

The data analyzed for this study are prospective secondary teachers' responses to prompts for teacher-created representations of practice included in the geometry materials. These materials take a transformation perspective, which is characterized by defining congruence and similarity via transformations (Usiskin & Coxford, 1972). The transformations critical to congruence and similarity are reflections, rotations, translations, dilations, and their compositions. The data come from the units on *Congruence Transformations* and *Similarity Transformations*. There were three units in total in these materials; a unit on *Axiomatic Systems* preceded the others. We did not include the representations of practice from *Axiomatic Systems* because they were written prior to the adoption of the writing standards.

MODULES geometry materials featured activities generating the definitions for each transformation type that emphasized the “logical necessity” (Simon, 2006) connecting the definition to the construction and identification of each transformation. The materials advised instructors to use these activities prior to assigning prompts for teacher-created representation of practice in each unit. As well, instructors were asked to have teachers discuss the student-level task in the representations of practice, and provided activities to do so. These activities were intended to help teachers notice features of high-quality responses to the student-level task prior to creating teacher-created representations of practice. To our knowledge, all instructors did so.



### 3.2 Prompts for Teacher-Created Representations of Practice in this Study

There were four prompts for creating representations of practice across the two transformation units. Two prompts asked prospective teachers to write narratives detailing how a classroom discussion might proceed, and two asked prospective teachers to video-record themselves as they would respond to the students whose work is provided. These prompts asked prospective teachers to envision a response to the students that would move students toward understanding connections between definitions and constructions of images of relevant transformations. Table 1 summarizes the prompts given to prospective teachers, and Figure 1 shows images of student work from some of the prompts.

**Table 1**

*Description of Prompts for Teacher-Created Representations of Practice*

Order	Medium	Response Format	Mathematical Content
1	Written	Written plan for class discussion	Constructing reflections
2	Video	Video response spoken to learners	Constructing rotations
3	Written	Written plan for class discussion	Distinguishing dilation from similarity
4	Video	Video response spoken to learners	Constructing dilations

**Figure 1**

*Sample secondary student work given to prospective teachers*

(a) As students are working on rotations of a flag, you observe two students with the following work completed.		(b) As students are working on constructing reflections, you observe two students with the following work completed.	
Student 1:	Student 2:	Student 1:	Student 2:

## 4. Data & Method

### 4.1 Overview

To develop a framework for characterizing the variation in Foundation and Contingency observed in teacher-created representations of practice, the authors analyzed teacher-created representations of practice in two rounds of coding. These efforts resulted in the operationalizations of Foundation and Contingency described earlier in Section 2.4 as well as the framework to be discussed later in this section.

### 4.2 Sampling

For this study, we selected data from eight cohorts across seven different teacher preparation programs in tertiary institutions, located in different regions of the US, that piloted the MODULES curriculum materials on geometry in one-semester content courses for prospective secondary mathematics teachers. The total dataset included more than 300 teacher-created representations of practice. The first round of coding used 75 teacher-created representations from 48 teachers ( $2 \text{ representations/teacher} \times 27 \text{ teachers} + 1 \text{ representation/teacher} \times 21 \text{ teachers}$ ) from the first year of data collection as well as initial data from the second year of data collection. The second round of coding used a purposive sample of 31 teacher-created representations of practice ( $4 \text{ representations} \times 7 \text{ teachers} + 3 \text{ representations} \times 1 \text{ teacher}$ ) drawn from the representations of 62 teachers with completed pre- and post-geometry assessments from the full set of data from the second year. There was an overlap of 1 teacher between data in the first round and second round. During the second round, we continued to refine operationalizations of Foundation and Contingency written during the first round of coding. We also continued to refer to the Knowledge Quartet website to ensure consistency in how we interpreted the constructs. Otherwise, we coded the representations from the single overlapping

teacher *de novo*. After second round coding concluded, we cross-checked second round and first round coding of the overlapping representations, and found that they were consistent for both Foundation and Contingency. The rounds were sequential: there were 18 months between the conclusion of the first round and the beginning of the second round.

We designed the purposive sample using a similar rationale to Rowland's (2013) selection for developing Knowledge Quartet codes: to document the range of potential knowledge use. Rowland (2013) used teachers' performance on an assessment of their mathematical knowledge as a proxy for teachers' knowledge in teaching, and selected a sample that "reflected the range of outcomes" (p. 19). Analogously, the purposive sample for this study was determined using performance on a subset of the GAST instrument (Mohr-Schroeder et al., 2017). We now give details about the GAST, then how we used GAST pre/post results to determine sampling. We blinded ourselves to teachers' GAST grouping and scores until after we completed coding. None of us had scored the pre-/post-tests. A scorer collated samples for us without disclosing scores.

**GAST Instrument.** There are 26 questions in total on the original GAST. We consulted our project's advisory board, which included Mohr-Schroeder, the lead researcher for GAST development, about using only questions from the instrument addressing content in the MODULES geometry materials. Based on this consultation, our pre-/post-forms contained 14 questions, with a total score of 22. Among the teachers whose data we collected for the second round, 61 had completed both pre- and post-semester GAST forms. The maximum pre-assessment score was 14, and the maximum post-assessment score was 17.

As context for interpreting the GAST scores, we note that the original GAST forms were validated to measure mathematical knowledge for teaching geometry using a sample of predominantly practicing teachers. Our sample was strictly prospective teachers. The mean score

on the full GAST form was 20 points out of a maximum of 30 (Mohr-Schroeder et al., 2017), and so our sample's maximum score may be comparable to their average score. Mohr-Schroeder et al. did not compare prospective and practicing teacher performance in their sample. However, using a different instrument to measure mathematical knowledge for teaching geometry, Herbst and colleagues reported that practicing secondary teachers score, on average, higher than preservice teachers do. Moreover, practicing teachers with more experience teaching geometry courses perform better than those with less experience (Herbst & Kosko, 2012; Milewski et al., 2019). Milewski et al. (2019) reported that most prospective teachers' scores were comparable to those in the lower half of practicing teachers' scores. It is not surprising that the teachers in our sample scored, overall, in a lower range than those in Mohr-Schoeder et al.'s sample.

**Sampling using GAST.** To document the range of potential knowledge use in teaching, we assigned "high performance" and "low performance" thresholds for each item on the GAST. GAST items used in this study were scored out of 1, 2, or 4 points. There thresholds were as follows. For 1 point items, "high performance" was 1 point; for 2 point items, "high performance" was 2 points; and for 4 point items, "high performance" was 3 or 4 points. Otherwise the score is categorized as "low performance".

We computed the percentage of "high-pre-score"- "high-post-score" pairs and "low-pre-score"- "low-post-score" pairs. In theory, someone's GAST responses could receive 100% "high-pre-score"- "high-post-score" if, on each item, their responses had a "high performance" score in both pre- and post-tests. Someone's GAST responses could also theoretically receive 100% "low-pre-score"- "low-post-score" if, on each item, their responses had a "low performance" score in both pre- and post-tests. In reality, most teacher received a mixture of all combinations across items: high/high, low/low, low/high, and occasionally high/low.

We narrowed the pool to 20 teachers (10+10 teachers), consisting of:

- The teachers with the 10 highest percentages of “high-pre-score”-“high-post-score” item scores, and
- The teachers with the 10 highest percentages of “low-pre-score”-“low-post-score” item scores.

In the first group, high/high percentages ranged from 45% to 73%, and low/low percentages ranged from 0% to 45.4%. In the second group, high/high percentages ranged from 0% to 27%, and low/low percentages ranged from 60.0% to 83.3%. Across the larger pool from which this purposive sample was drawn, high/high percentages ranged from was 0% to 72.7% and low/low percentages ranged 9.1% to 83.3%.

We continued analysis with the 8 teachers out of the 20 who had submitted all 4 representations of practice selected for this analysis. See Table 2. During analysis we realized that one assignment had been scanned incompletely. At this point in the study, we no longer had access to contact the teacher and thus eliminated this one representation from our sample. This resulted in a sample of 31 teacher-created representations of practice.

**Table 2**

*Purposive sampling of teachers ordered by “high-pre-score”- “high-post-score” item percentage*

teacher	high-high proportion*	low-low proportion**	GAST (Pre)	GAST (Post)	GAST (Difference)
GMM205	45%	27%	9	12	+3
GMM202	45%	27%	7	11	+4
GTA218	45%	36%	9	10	+1
GMM308	27%	64%	5	9	+4
GMM302	18%	64%	6	9	+3
GTA208	18%	73%	2	3	+1
GMM206	17%	83%	1	6	+5
GMM201	11%	67%	3	9	+6

\* high-high proportion = proportion of “high-pre-score”-“high-post-score” GAST item percentage.

\*\* low-low proportion = proportion “low-pre-score”-“low-post-score” GAST item percentage

### 4.3 Developing a Framework to Characterize Foundation and Contingency

To develop a framework for characterizing foundation and contingency, the authors first considered Rowland and colleagues' descriptions of the Foundation and Contingency dimensions of the Knowledge Quartet (e.g., Rowland et al., 2016). We reflected on how these descriptions may apply to the specific teacher-created representations of practice analyzed and, at the same time, how they may apply to other domains. To do so, the authors involved researchers with expertise in a variety of mathematical domains—such as mathematical modeling, algebra, and geometry—in our discussion.

We emphasize that our codes throughout the study represent *demonstrated* use of mathematical knowledge. We do not claim that this is the totality of the teachers' knowledge; the knowledge may have been tacit but possessed. It is therefore incorrect to extrapolate about the entire body of a teacher's knowledge based on these representations of practice. However, these data may still give some insight into how mathematical knowledge use is expressed in the context of a university content course.

We determined coding in both rounds by consensus. We only finalized a code when all coders agreed on the code assigned, as well as the reasoning. The reasoning was then incorporated into the operationalizations used, and all codes made to that point were checked against the revised operationalization.

In our first round of coding, we were able to reconcile most but not all differences for Foundation codes across four levels. However, efforts to agree on distinctions among four levels of Contingency observations reached no conclusion.

In the second round of coding, for both Foundation and Contingency, we used two categories of “High” and “Developing”. We anticipated that having two categories would result in greater

reliability among coders, because we were least able to draw clear distinctions in the first round on the middle and lowest categories. We began by coding demonstrated Foundation. In contrast to the first round, where nearly half the codes needed reconciliation, only 4 codes for 31 representations needed reconciling. We repeated this process for demonstrated Contingency. Contingency coding required reconciling 12 codes across the 31 representations.

## **5. Results**

Our primary result is a framework for characterizing demonstrated high and developing knowledge in the Foundation and Contingency dimensions in teacher-created representations of practice created in the context of a university mathematics course. Figure 2(a) shows this framework across domains. Figure 2(b) specializes the framework to definitions of geometric transformations. Figure 3 shows assigned codes for all representations of practice in the purposive sample. Table 3 shows the four combinations of High and Developing codes.

**Figure 2***Framework for Characterizing Foundation and Contingency Knowledge*









(a) Across domains		
	FOUNDATION	CONTINGENCY
HIGH	Recognizes logical necessities in patterns and relationships in the intended mathematics.	Frames questions or explanations about patterns and relationships in terms of given student thinking.
DEVP	Does not explain patterns or relationships in the intended mathematics.	Superficial use of student thinking in explanations of patterns or relationships.



  

(b) Within definitions of transformations, in geometry		
	FOUNDATION	CONTINGENCY
HIGH	<p>Recognizes the logical necessity connecting the definition of a transformation to ways of constructing a transformation image</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Explains a method of construction by marking points on a preimage and then “applying the definition to each of the marked points” to obtain the image</li> <li>Reasons that an attempted image is incorrect by showing that it does not satisfy the transformation definition</li> </ul>	<p>Frames questions or explanations about connection between construction and definition in terms of given students’ thinking</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Directs attention to student work to understand the idea that all properties of the definition must be followed to produce a correct construction</li> <li>Engages students in selecting locations in sample student work, and reasoning whether the definition is satisfied</li> </ul>
DEVP	<p>Explicitly or implicitly treats the definition of a transformation as separate from constructing images, and/or demonstrates lack of understanding of definition</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Describes a method for constructing, and never mentions any definition</li> <li>Provides incorrect definition</li> </ul>	<p>Superficial use of student thinking in explanation of connection between construction and definition</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Evaluates student work as “right” or “wrong”; does not cite work otherwise</li> <li>Provides a correct explanation that does not reference student work</li> <li>Only uses own extrapolations of student thinking, and does not incorporate student contributions provided in prompts for creating representation of practice</li> </ul>



**Figure 3***Codes for observed knowledge use in purposive sample*

Teacher	Foundation				Contingency				#H
									
GMM 205	H D	H H	H D	H H	4	2			
GMM 202	H H	H H	H H	D D	3	3			
GTA 218	D D	D D	D D	D D	0	0			
GMM 302	D D	H H	D D	D H	1	2			
GMM 308	H D	D H	H H	D H	2	3			
GMM 201	D D	D D	D D	D D	0	0			
GTA 208	D D	D D	D D	—	0	0			
GTA 206	D D	H D	D D	D D	1	0			

 = video representation practice  
 = written representation of practice  
 Representations of practice listed in order of appearance in curriculum  
 H = High, D = Developing  
 # H = total High codes in dimension

**Table 3***Foundation/Contingency code combinations*

Combination Foundation/Contingency	Teachers whose representations of practice were coded with this combination (how many)
High F/High C	GMM205(2), GMM202(3), GMM302(1), GMM308(1)
Developing F/High C	GMM302(1), GMM308(2)
High F/Developing C	GMM205(2), GTA206(1)
Developing F/Developing C	GMM202(1), GTA218(4), GMM302(2), GMM201(4), GTA208(3), GTA206(3)

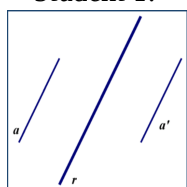
We now describe the four combinations of High or Developing Foundation and Contingency with selected teacher-created representations of practice. To do so, we take advantage of the fact that there is one representation of practice that showcases all four combinations. This is the written prompt in Figure 3, Column 2. Figure 4 shows the full text of this prompt. Note that the student work is similar to that in Figure 1. Figure 5 shows a definition for reflection suggested in the curriculum materials.

**Figure 4**

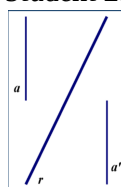
*Summary of prompt given to teachers to create a written representation of practice*

Students in your 9th grade math class have defined the isometric transformations and are working on performing transformations in preparation for exploration of the properties of the transformations. As students are working on reflections of a segment across a line, you observe two students with the following work completed.

Student 1:



Student 2:



**Write a paper** in which you clearly describe a plan for how you will conduct a whole class discussion which will allow you to elicit student thinking about these reflections, with specific use of the two example students' work, and move the class toward understanding connections between methods of reflection and the definition of reflection. Your plan for the discussion should include discussion questions, descriptions of the ways you anticipate that students might respond to questions, and any appropriate tasks that will move student thinking forward. Your response should indicate your understandings about the definition of reflection and ways in which this is applied to various methods of construction.

**Figure 5**

*Version of class definition of reflection*

A reflection across a line  $L$  is a transformation that, for every point  $P$  in the plane:

- $P' = P$  if  $P$  is on  $L$
- $L$  is the perpendicular bisector of segment  $PP'$  if  $P$  is not on  $L$ .

*Note: These materials use the convention that  $P'$  refers to the image of a preimage  $P$  under the transformation discussed.*

## 5.1 High Foundation/High Contingency

Before we begin, we remind the reader that all illustrations are for the written prompt shown in Figure 4. While we ground our descriptions in details of the chosen prompt, we have also highlighted aspects that generalize across the category. After each specific description, we step back and summarize the signal qualities for each category.

In our framework, the quality of demonstrated Foundation is characterized by linking constructions to the definition, and the quality of Contingency is characterized by integrating student work into the work of connecting constructions and definitions. GMM302's representation of practice was coded as High Foundation/High Contingency. GMM302 began their representation of practice:

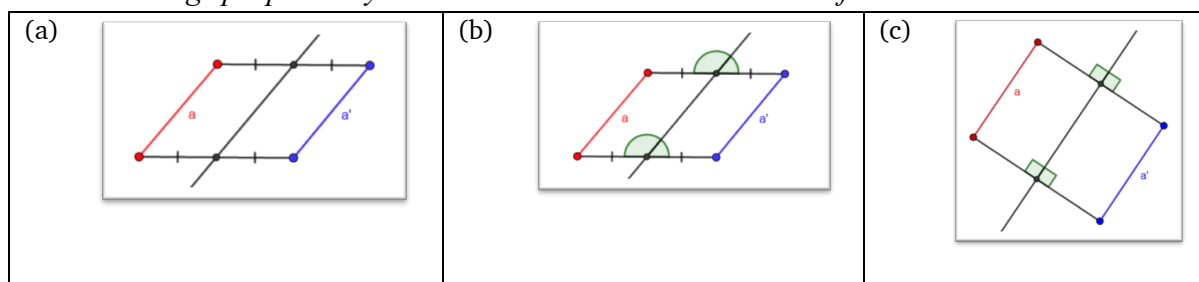
To start, I would draw the student responses and our definition of Reflection on the board. [...] Pointing to the first response, [I would ask,] if we were to draw a line between the points  $P$  and the corresponding  $P'$ 's, what can we tell about the line segments made by  $P$  and  $P'$ ? As students respond, I draw and make the corresponding changes to the figure on the board (see Figure 6(a)).

Then, after describing some potential responses from students, GMM302 prompted students to link construction and definition: "What is it we know about our line of reflection in regard to our definition of reflection?" GMM302 then marked the angles (see Figure 6(b)), asked students questions to review the two defining properties of perpendicular bisectors (bisecting, and with perpendicular angles), and posed the question, "Since our main problem here is the angles, how might we approach this in a way that results in right angles instead?" Finally, after drawing a correct reflection (see Figure 6(c)) but without evaluating it as such to the students, GMM302 asked, "Looking at our new figure, does this hold true to the definition of a Reflection?"

GMM302 concluded, after describing potential responses, “Yes, it does hold true. So, we know [segment]  $a'$  is the reflection of [segment]  $a$  across the given line.”

**Figure 6**

*Board drawings proposed by GMM302 to link construction to definition*



We coded this representation as High Foundation/High Contingency because GMM302 created tight connections from incorrect and correct images to the definition, positioned students to engage with these links, and did so while centering student work. Other examples that were coded as High Foundation/High Contingency also exhibited close connections between the definition of the transformation and its construction. In each case, the definition was used as a tool to “check” the correctness of the provided student work (as in the case of GMM 302) or to develop explanations through questioning. For example, a teacher might provide an image and ask questions of the class relating to each portion of the definition of the related transformation, either with specific mention of the definition by the teacher or by asking questions that would lead to the definition being provided by a student in the class.

## 5.2 Developing Foundation/High Contingency

GMM308’s response was coded Developing Foundation/High Contingency. GMM308 began their representation of practice by analyzing the student work, and suggesting what may have been going on in the students’ mind that led to these constructions.

It looks as though they have drawn lines across the line of reflection from each point to the reflected point. I believe that they have thought that since it is reflected that the distance from the line of reflection is now opposite for each point (the point on top of the reflected image is the same distance as the point on the bottom of the pre-image and vice versa).

In this way, GMM308 exemplified the notion of interacting reflectively with the given student thinking (Thompson, 2000). GMM308 then described linking the work to the definition:

I would use [Student 2's work] to discuss with students how this attending to some points of the definition, but not quite (sic). They have used the idea of the same distance from the line of reflection, but it was utilized incorrectly. I would use this to be able to discuss with students how this doesn't fully fit the definition of a reflection and how we can fix that. We would work as a class to improve the original reflection and make sure it fits all of the necessary components of the definition needed.

This representation of practice exemplified High Contingency: it identified specific connections *from given student work* to the definition of reflection, including the role of perpendicular bisectors. However, it did not articulate the reasoning about perpendicular bisectors precisely, and so we assigned the Foundation code as Developing. This was the case for GMM308's presented analysis of both provided samples of student work.

This pattern was repeated in each representation coded Developing Foundation/High Contingency. In each, the response begins by drawing out the thinking provided by one of the students related to the appropriate transformation. Then, the responses envision questions to students where pieces of the definition for the transformation are provided but not connected to the construction of the transformation. By starting with the given student work, these responses

demonstrate High Contingency, but unclear or inaccurate connection between the definition and the construction of the transformation leads the response to be coded with Developing Foundation. In our data, there were two ways for a connection to be coded as Developing Foundation with High Contingency. First, as demonstrated by GMM308, the reasoning about the definition is imprecise. Second, a teacher may reason about the definition incorrectly.

### **5.3 High Foundation/Developing Contingency**

In our framework, demonstrating High Foundation requires clearly linking the construction to the definition whereas demonstrating High Contingency relies on the use of student work to make the connection between the definition and construction. GTA 206's representation of practice was coded High Foundation/Developing Contingency.

This representation of practice began by posing to the class, "What do we know about reflections?" and encouraging discussion specifically focused on the definition of reflection. GTA 206's representation then proposed student-generated definitions that were correct but imprecise, and how they would clarify the role of the perpendicular bisector in the definition:

I will discuss this by showing students different examples on the board on how this comes about by drawing a perpendicular bisector between point A on Object 1 and point A' on Object 2. I will do this with each point of the object to show that the perpendicular bisector does in fact make a right angle at the line of reflection. Doing so links the definition to the way in which one can determine whether or not two figures are in fact a reflection of each other.

At this point, GTA206's representation of practice turned to more examples of student work but did not choose to use the student work provided: "I will at this point use student 1 and

student 2's examples, but not the exact same lines as the students drew. I will change them so that they do not feel as if I am targeting their work . . ." GTA206's representation of practice suggests that learners will be able to determine their errors by examining other work that is not their own. The discussion of reflections concludes with an assignment in which learners would "draw a perpendicular bisector from each original figure point to the new reflected figures."

GTA206's representation of practice made a clear connection between the definition and the construction of the reflection, exemplifying High Contingency. However, GTA206's representation of practice did not draw on the provided learner thinking in moving the class forward in their understanding of reflections. This representation of practice provides no evidence of knowledge for responding to potentially unexpected contributions from students, because they did not reference the given student contributions at all. Perhaps GTA206 left their knowledge of the connection tacit rather than expressed, because they prioritized the students' emotions. Although consideration for the learner's feelings as their work is shown is important, it is also vital to use the thinking of the students in the class in order to move their thinking forward. We coded this response as Developing Contingency, but also use this case to remind us that we are coding only proxies of knowledge.

The other examples in our sample that were coded as High Foundation/Developing Contingency were video responses. In each case, the teachers correctly described the connections between the provided student work and definitions of the appropriate transformations, demonstrating High Foundation as they evaluated the student work. In their descriptions of how they would respond to the students, they each directed students to new examples rather than the provided work and related these new examples to the definition. In this way, the teachers did not connect responding to the student with the given student work.

#### 5.4 Developing Foundation/Developing Contingency

A response that demonstrates Developing Foundation and Developing Contingency lacks connection between definition and construction, and furthermore does not use the given student work to address the concepts. GMM201's representation of practice illustrates this type.

GMM201's representation of practice opened by sharing the two student samples and asking students "if they agree or disagree with the two works." The representation stated that GMM201 intended to probe students for their reasoning, and give time to share their responses to the whole class. Although potential student responses are given that include reference to the distance of a point from the line of reflection, there is no resolution or clear use of the definition of reflection.

Rather than resolving this discussion, GMM201's representation of practice then posed a new task to students, where the students were asked to reflect provided shapes over a given line. Then, after monitoring students' work on this task, the representation of practice stated that the "teacher will provide the definition of reflection to the students, and work on another example on the board to show how the definition is related to the work." This is the only mention of a definition of reflection in GMM201's response. Note that there was no elaboration of what the definition is or how it is used to construct a reflection. GMM201's representation concluded by indicating that students will then be asked to revise their answers to previous problems and to determine in other ways, such as paper folding, whether or not their images are reflections.

At first read, it may seem that GMM201 used student thinking by displaying student responses and providing time for learners to make sense of the work they observe. However, this representation of practice never linked the given student work to the definition of reflection, and also makes no substantive reference to the given student work beyond displaying it. For these reasons, we coded this representation of practice as Developing Foundation/Developing



Contingency.

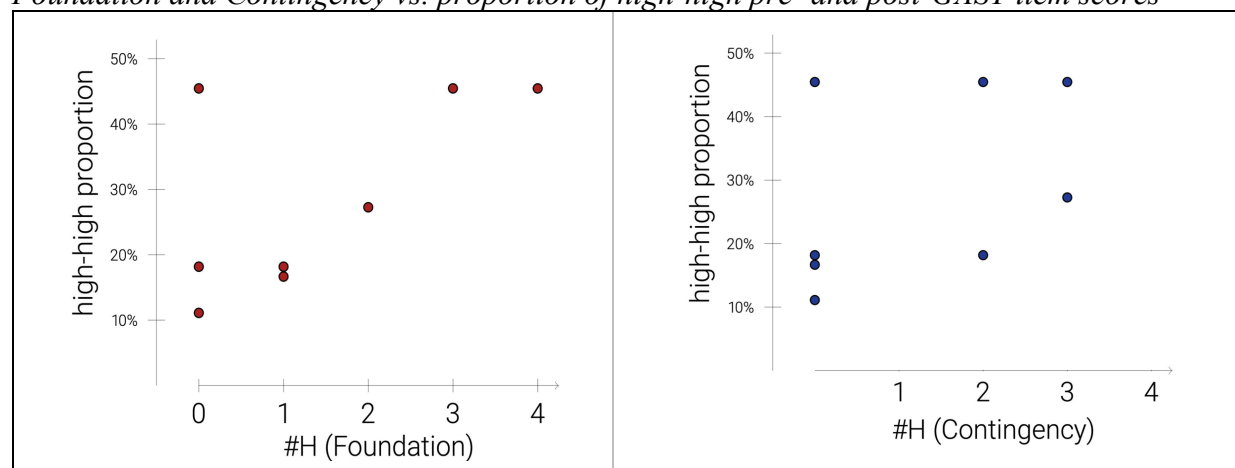
Other examples coded in the Developing Foundation/Developing Contingency category similarly lacked substantive use of student work and explicit connection between definition and construction. The student work is presented at some point during the representation of practice but it is not taken up beyond asking whether or not the class agrees with the work, if at all. Responses in this category did not build discussion from the initial prompts asking for agreement. Some representations did state the definitions of the respective transformations. The definition was either not given, or stated by the teacher with no connection to the student work samples. Students were directed to a new task, without connection between these pieces.

### 5.5 Foundation and Contingency levels in relation to GAST selection process

Recall that we selected the purposive sample in part by the proportion of “high-pre-score”- “high-post-score” GAST item percentages (shown in Table 2). We coded Foundation and Contingency for these teachers’ representations of practice. To consider the relation between our sampling procedure and the coding, we plotted their item percentage in relation to the number of High Foundation and High Contingency codes for each teacher. See Figure 7.

**Figure 7**

*Foundation and Contingency vs. proportion of high-high pre- and post-GAST item scores*



Overall, as observed in Figure 7, the proportion of “high-pre-score”-“high-post-score” GAST item percentages appear to associate more strongly with Foundation codes than Contingency codes. This may be explained by the content of the GAST. When we analyzed GAST items administered to teachers for whether they fit the description of knowledge in the Foundation or Contingency dimensions, we only found 1 item to fit the description of Contingency: the question asked teachers to incorporate given student work. All other items assessed knowledge of particular mathematical theorems in the form of a purely mathematical question, or involved Foundation knowledge to analyze a proposed mathematical task.

Among the teachers with the highest “high-pre-score”-“high-post-score” percentages (GMM 205, GMM 202, GTA 218; see Table 2), two exhibited high levels in the foundation and contingency dimensions on at least half of their teacher-created representations of practice. Overall, their representations of practice demonstrated greater knowledge use in Foundation and Contingency than the representations of practice from teachers selected for the highest “low-pre-score”-“low-post-score” percentages. The exceptions are GTA 218 and GMM 308. GTA 218 only improved by one point on their post-test GAST performance; perhaps this indicates an overall difficulty with developing deeper mathematical knowledge overall in the course. This would explain a difficulty applying mathematical knowledge to the work of teaching at high levels in the representations of practice that they created, particularly if this semester was their first opportunity to apply mathematical knowledge directly to the work of teaching. Another explanation is that GTA 218 did not engage with this material, out of a lack of interest, pressures outside of class, or other reasons.

Across all four of GMM 308’s representations, two High Foundation and three High Contingency codes were assigned. As noted in the previous discussion of GMM 308’s

representations of practice, there is attention to potential connections between definition and construction. Developing codes in Foundation dimension were given when the connection was not precisely articulated. In other words, GMM 308's representations of practice demonstrated an attention to underlying mathematical structures, but this attention was not always expressed as precisely as needed to clearly connect the construction to the definition and thus to be coded as High Foundation.

Representations from the three teachers with the highest “low-pre-score”-“low-post-score” percentages were coded with only Developing Contingency and Foundation. Two of these teachers' GAST scores saw the largest increase from pre- to post-test across the sample. These teachers were developing mathematical knowledge for teaching from the most basic levels, may have had difficulty applying their knowledge to teaching, despite their knowledge gain.

## 6. Discussion

We set out to examine the Foundation and Contingency knowledge observable in teacher-created representations of practice, and to characterize their variation. To do so, we analyzed a purposive sample of teacher-created representations of practice in geometry. We now address limitations of the analysis and the potential generality of the framework.

First, one limitation to coding any proxy of knowledge is that “Developing” codes may reflect the absence of evidence rather than absence of knowledge. Developing Foundation or Contingency may be related more to prioritizing different aims, such as student affect or a desire for brevity. It may also come from disengagement with the course. Yet we also observe that we collected data from courses where teachers generated definitions for transformations based on constructing images, routinely analyzed and responded to sample secondary student thinking, and discussed featured secondary student-level tasks in view of the connection to definition.

These activities were offered prior to assigning the prompt for the representations of practice. Considering these curricular supports, we conclude that at least some Developing Foundation and Contingency codes indicate where a teacher's knowledge can grow, although we cannot absolutely conclude that for all responses.

Second, we would be remiss not to issue caveats about the binary scheme of “Developing” and “High”. These characterizations, like other hierarchical characterizations in the literature (e.g., Ader & Carlson, 2021; Munter, 2014; Serbin et al., 2020), are not intended to be characterizations of teachers or their ultimate potential. Rather, we present descriptions of observable features of representations of practice that point to mathematical and pedagogical areas teachers may need more support to develop, and also to showcase how exemplary knowledge may manifest.

In evaluating the robustness of our framework, we consider the limitation of our data to four prompts for representations of practice in geometry, and the potential for our framework to generalize across domains. Our framework, as reported, is tailored to the use of definition to a particular concept of geometry, and derived from the analysis of a limited number of prompts. However, we see our framework as generalizable. Its underpinnings in the Knowledge Quartet (Rowland, 2013) and mathematical understanding (Simon, 2006) are intended to apply broadly to mathematics teaching and learning. The centrality of definition to mathematics, as well as reasoning with definition and assumptions (Kitcher, 1984), suggests the potential for adapting this framework to domains with strongly structured logical systems, such as algebra.

For instance, in place of linking definition with construction methods, the framework could emphasize connecting definitions with common procedures. As an example, consider inverse functions. Multiple studies have shown that students compartmentalize the various meanings and

representations of inverse functions, not understanding how one is related to the other, or to definitions. Students may not see connections between “switching-and-solving”, reflecting a graph about the line  $y = x$ , and the definition of inverse function (see Paoletti, Stevens, Hobson, Moore, & LaForest, 2018 for a review). Yet we would hope that teachers would understand these connections, and moreover, be able to apply knowledge of this connection to their teaching (e.g., Weber et al., 2020). In Figure 8, we interpret this situation in terms of our framework.

**Figure 8**

*Potential generalization of framework to inverse functions*

	FOUNDATION	CONTINGENCY
HIGH	<p>Recognizes the logical necessity of connecting the definition of inverse function to ways of constructing inverse functions, representing functions and their inverses, and determining whether a proposed function is the inverse of a given function.</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Relates a method of construction to the definition of inverse function.</li> <li>Explains why different conceptions of inverse function (e.g., “reverse of the function process”, “switching x and y and solving for y”; see Vidakovic, 1996) are equivalent, in terms of the definition of inverse function.</li> </ul>	<p>Frames questions or explanations about connections among representations, constructions, and reasoning about inverse functions in terms of given students’ thinking.</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Directs attention to various given student conceptions and connects them via the definition of inverse function.</li> <li>Engages students in comparing behavior evident in students’ representations of a function and its inverse, and why these illustrate the defined relationship between an invertible function and its inverse.</li> </ul>
DEVP	<p>Explicitly or implicitly treats the definition of inverse function as separate from representing or constructing an inverse function, and/or demonstrates lack of understanding of the definition.</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Describes how to construct an inverse strictly procedurally, without mention of a definition (cf. Brown &amp; Reynolds, 2007).</li> <li>Provides imprecise definition of inverse function.</li> </ul>	<p>Superficial use of student thinking in explanations of connection between definition and representations, constructions, or other reasoning.</p> <p><i>Examples:</i></p> <ul style="list-style-type: none"> <li>Evaluates student work as “right” or “wrong”; does not cite work otherwise</li> <li>Provides a correct explanation that does not reference student work</li> </ul>

For domains such as mathematical modeling, which apply mathematics in phases of distinctive practices (e.g., Blum & Leiß, 2007), the framework could emphasize the rationale for

each phase as well as anticipation of movement across phases, for instance, knowing that the proposal of a mathematical model can be followed by considering the real world or the results of the model, that that these phases can work together to refine one's model (e.g., Czoher, 2018).

Finally, we note that our case for Foundation is stronger than our case for Contingency: we have applied the framework to more data for Foundation, due to the first round coding process. Nonetheless we believe that our case for Contingency holds promise, and that the specific attention to given student work is a contribution to the field.

## 7. Conclusion

We now consider our results with respect to the literature on teachers' knowledge. We see Foundation knowledge as content knowledge, pedagogical content knowledge as elicited by applications of mathematics to teaching, and Contingency as an outcome of pedagogical content knowledge. Using the Knowledge Quartet allowed us to attend precisely to how mathematics can be applied to teaching when designing prompts for teacher-created representations of practice.

As Baumert et al. (2010) noted, "one of the next great challenges for teacher research will be to determine how [content knowledge and pedagogical content knowledge] can be best conveyed" (p. 168). Teacher education must address both content knowledge and pedagogical content knowledge, and in ways that connect to teaching. Rowland (2013) posited that Foundation knowledge determines the use of mathematics in the other dimensions of the Knowledge Quartet, including Contingency.

Our work provides a proof of concept of a device that can be used in multiple university mathematics courses to integrate mathematics and its application to teaching. The representations of practice we analyzed were created by teachers as part of a suite of university mathematics courses. In the representations studied, they elicited Foundation knowledge that could be

integrated with Contingency actions. We believe that these results extend across content areas.

Our results speak to Monk's (1994) argument for a more nuanced understanding of how mathematical knowledge may contribute to secondary teaching. We find that mathematical knowledge, such as can be taught in a university mathematics course, can be observed while explicit connections to teaching are made. Moreover, we demonstrated ways that quality in two dimensions, Foundation and Contingency, may be articulated.

Our data corroborates Rowland's posited relationship between Foundation and Contingency: High Foundation tended to correspond with High Contingency, and Developing Foundation tended to correspond with Developing Contingency. However, a High Foundation code was not a guarantee of High Contingency. We do not have the data to know whether our results corroborate Baumert et al.'s (2010) findings for content knowledge as a precursor of pedagogical content knowledge.

Our results also indicate how far we must go to meet Baumert et al.'s (2010) challenge. There were not many High Foundation/High Contingency codes, and producing a representation of practice coded as High earlier in the term did not correspond to being coded High later in the term. Moreover, there were many Developing/Developing code combinations. One interpretation is that, like Herbst and colleagues' findings with the MKT-G instrument, there is on average a wide gap between prospective teachers' and practicing teachers' knowledge for teaching geometry. Another interpretation is that it is difficult to transfer mathematical teaching practices learned in one mathematical context (e.g., congruence) to another (e.g., similarity); a similar result was found at the elementary level by Morris et al. (2009). Whatever the explanation, the challenge remains for connecting mathematics to its applications in teaching, and in ways that support teachers' development across content areas.

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