Mathematics courses for teachers focusing on a coherent, logical development of ideas connected to multiplication

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My principle 1

Mathematics, including school mathematics, is beautiful, deep, and richly interconnected at every level. Thinking deeply about school mathematics can be a joyful, challenging, and worthwhile intellectual experience.

My principle 2

Consider teaching mathematics courses for teachers that are coherent, logically structured, and derived from a few core ideas in elementary mathematics, and that also draw on and develop teachers' own reasoning and sensemaking.

Mathematics courses for future teachers

At UGA, Andrew Izsák and I developed courses for future middle-grades and secondary teachers that:

- Focus on topics connected to multiplication:
  division, fractions, ratio, proportion, and linear relationships
- Develop logical explanations that draw on future teachers' own reasoning and sensemaking
- Seek coherence

How to create coherence?

Fractions can be interpreted in different ways

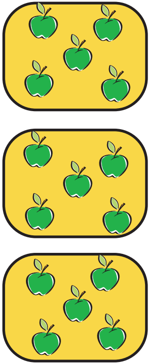
$$\frac{A}{B}$$

can mean:

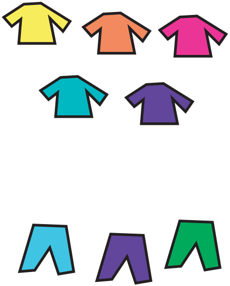
- A out of B equal parts
- the ratio A to B
- $A \div B$
- A point on a number line

How to create coherence?

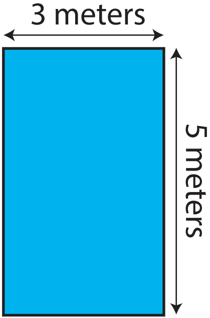
How many apples?



How many outfits?



How much area?



How to create coherence?

Why is multiplication a coherent concept

- across different types of word problems?
- across whole numbers and fractions?

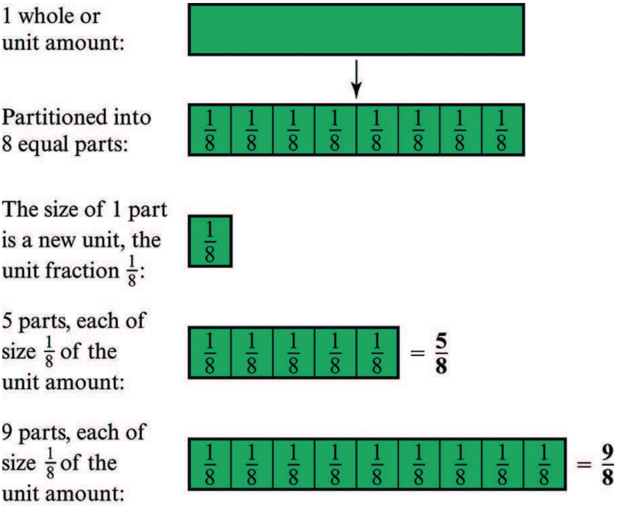
How to create coherence?

Our approach: weave the ideas together by returning repeatedly to

- a definition for fractions (the CCSS definition)
- a definition for multiplication

Both take a measurement perspective.

Defining fractions (Common Core definition)



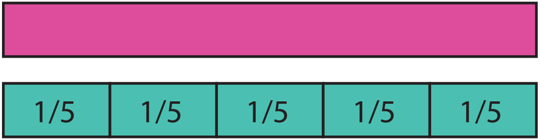
Interpreting fractions as results of measurement

How much of this unit

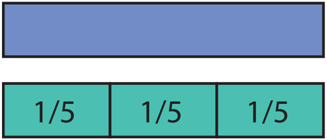
does it take to be equal to this strip?

Interpreting fractions as results of measurement

How much of this unit



does it take to be equal to this strip?



Answer: 3/5

3/5 means "3 fifths"
3 parts, each 1/5

A unified concept of number through measurement

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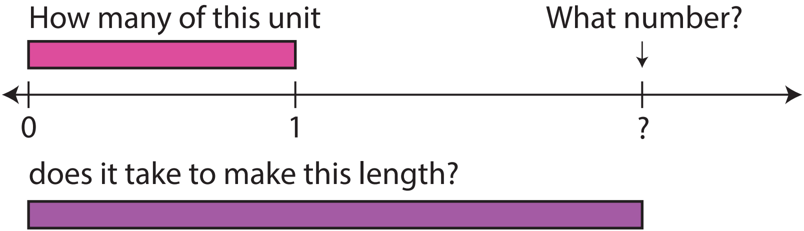

Multiplication and division through measurement

How many of this

does it take to be equal to this?

Multiplication and division through measurement

How many of this

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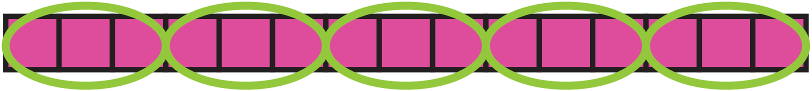


How many 3s does it take to equal 15?

$$? \cdot 3 = 15$$

Five 3s equal 15

$$5 \cdot 3 = 15$$

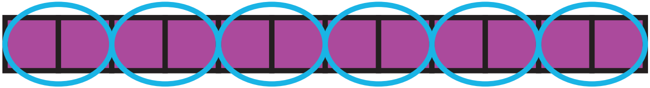
Multiplication and division through measurement

$$6 \cdot 2 = ?$$

Six 2s are equal to what number?

6 of this

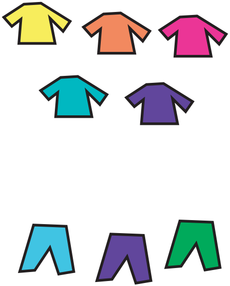
makes what number of blocks?



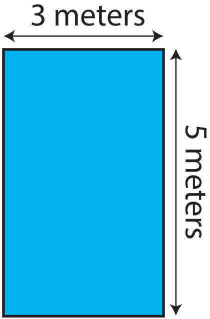
The multiplicand, 2, becomes a unit of measurement.
The multiplier, 6, tells us how many times to take that unit.

Why does multiplication apply?

How many outfits?



How much area?



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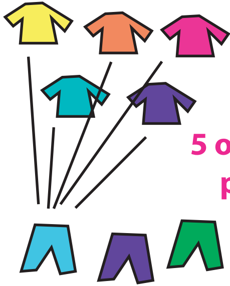
Why does multiplication apply?

How many outfits?



5 outfits for each pair of pants

3   5 outfits make how many outfits?

$3 \cdot 5 = ?$

outfits   outfits

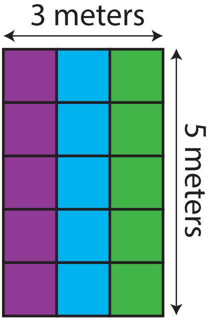
1 group = 5 outfits

1 group <---> 1 pants

Re-interpret 5 shirts as 5 outfits (for 1 pants)

Why does multiplication apply?

How much area?



3   5 sq meters make how many sq meters?

$3 \cdot 5 = ?$

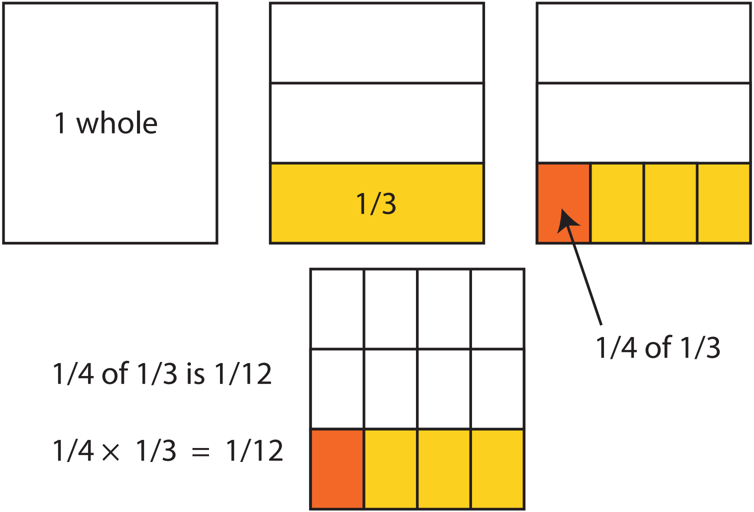
sq m   sq m

1 group = 5 sq meters

1 group <---> 1 meter width

Re-interpret 5 meters as 5 square meters (for 1 meter of width)

Fraction multiplication from a measurement perspective



A definition of multiplication based in measurement

$$M \cdot N = P$$

N base units: **1 group** measured in base units

P base units: **M groups** measured in base units

M of **this group** make P base units exactly.

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How to create coherence?

$$M \cdot N = P$$

Multiplication problems:

$$M \cdot N = ?$$

Division problems:

$$M \cdot ? = P \qquad ? \cdot N = P$$

Linear (proportional) relationships:

$$M \cdot ? = ? \qquad ? \cdot N = ?$$

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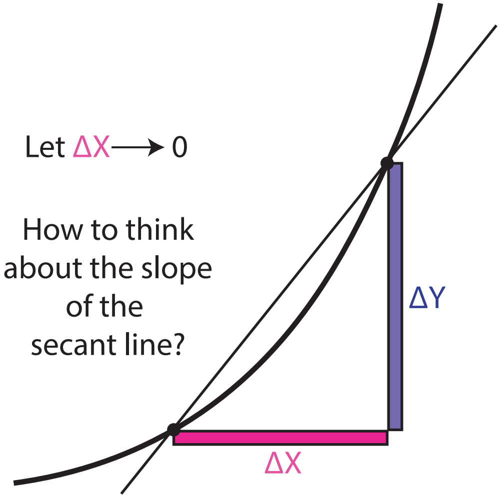
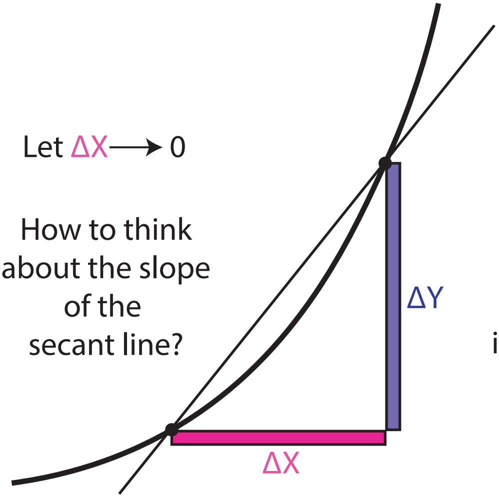
Looking ahead to calculus



A measurement interpretation of rate and slope

Let $\Delta X \longrightarrow 0$

How to think
about the slope
of the
secant line?



Slope =
how many
ΔXs
it takes to make
ΔY
exactly

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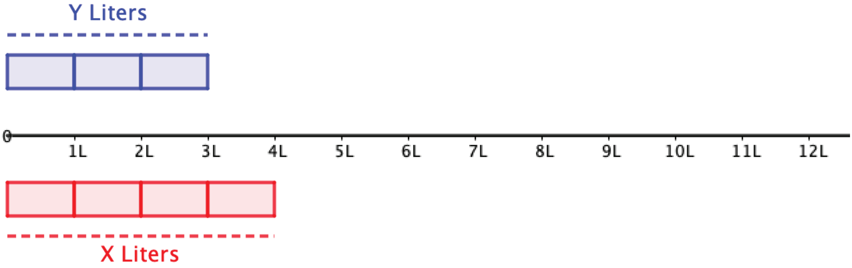
Context: A fixed purple hue made from red and blue



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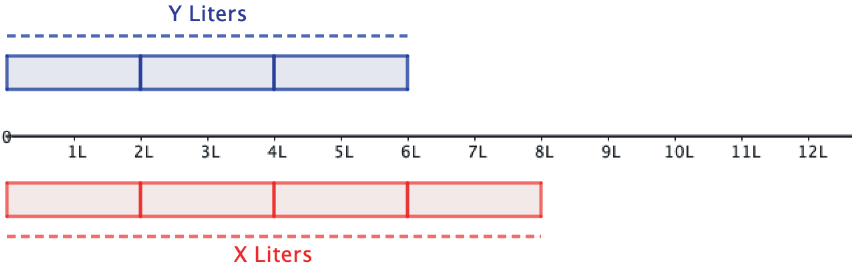

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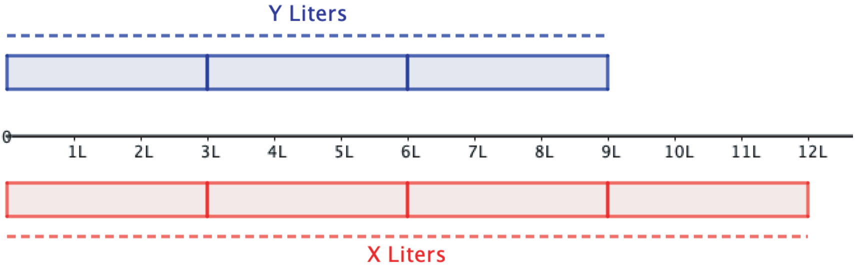

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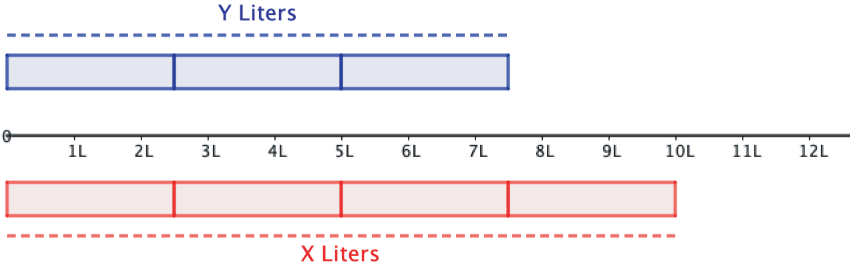

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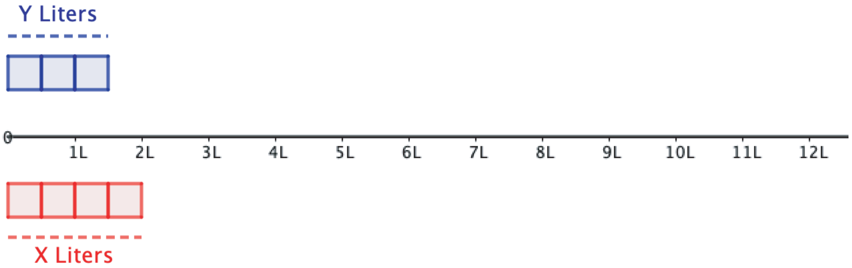
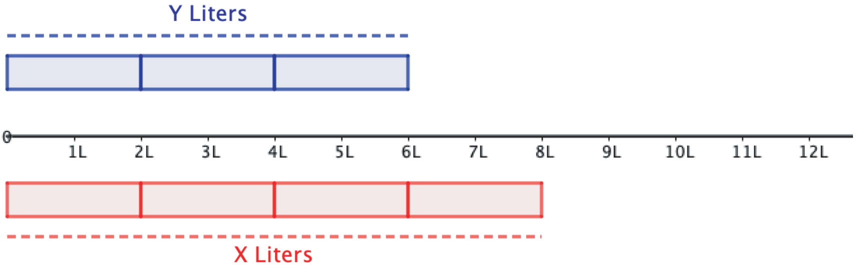


A measurement interpretation of rate and slope

Use the volume of red paint as measurement unit:
The volume of blue paint is always $\frac{3}{4}$ the volume of the red paint.



Y is always $\frac{3}{4}$ of X

$Y = \frac{3}{4}X$

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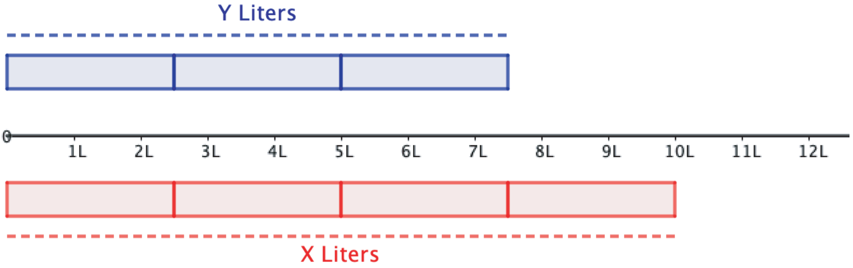

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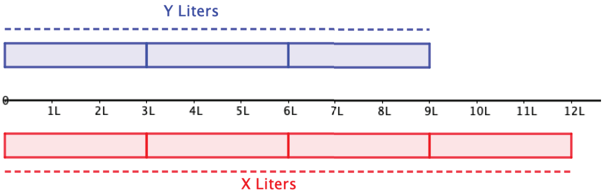

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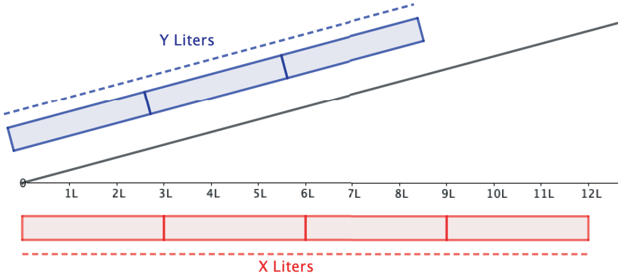

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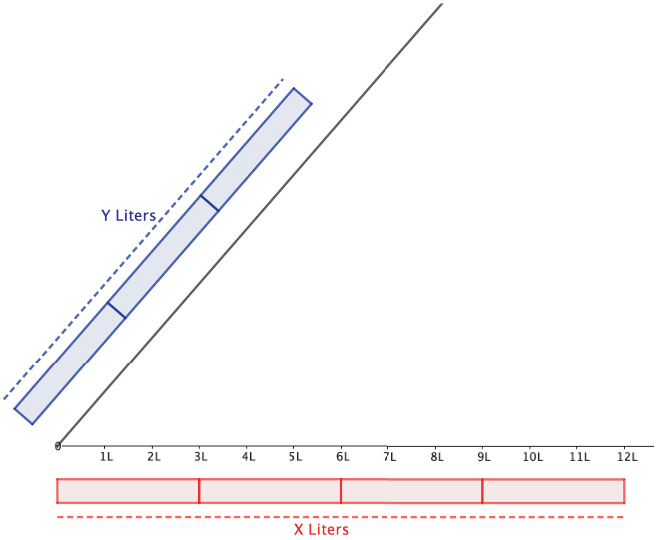

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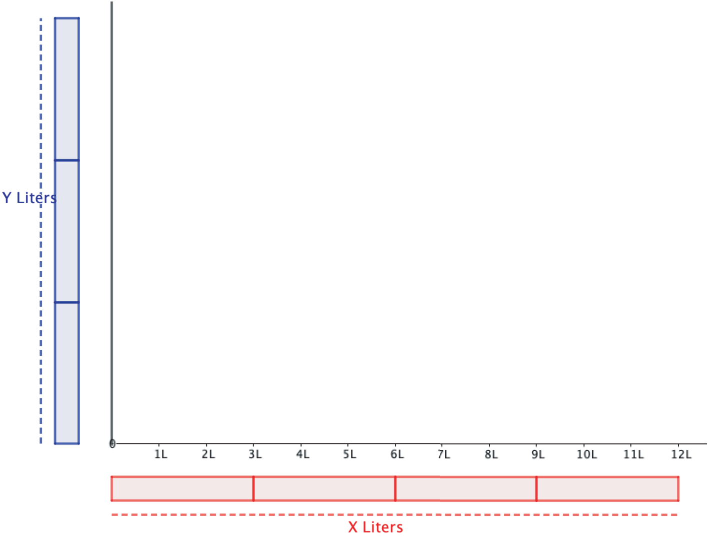

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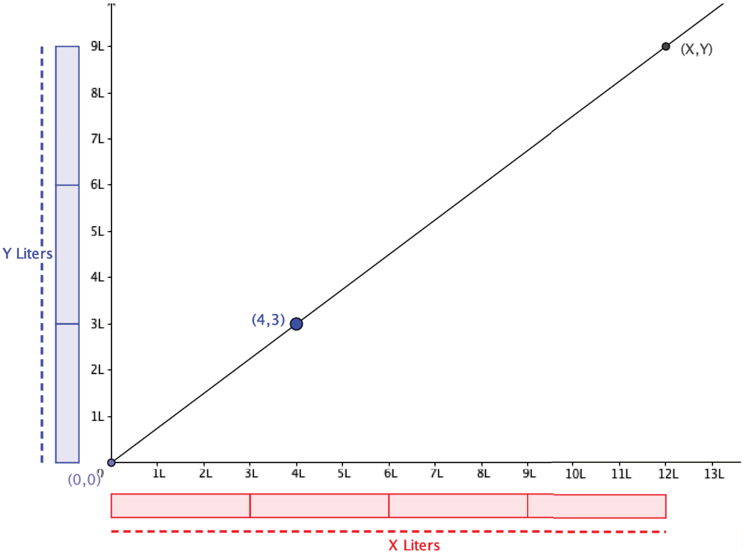
The *variable-parts* perspective on lines



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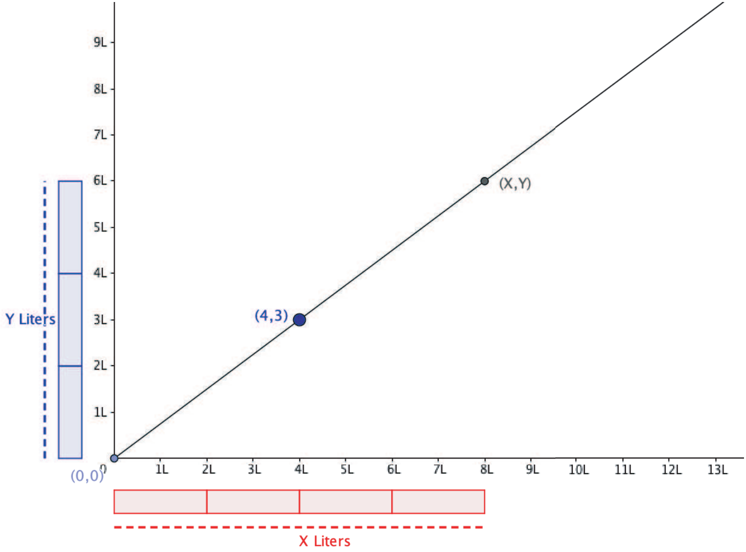

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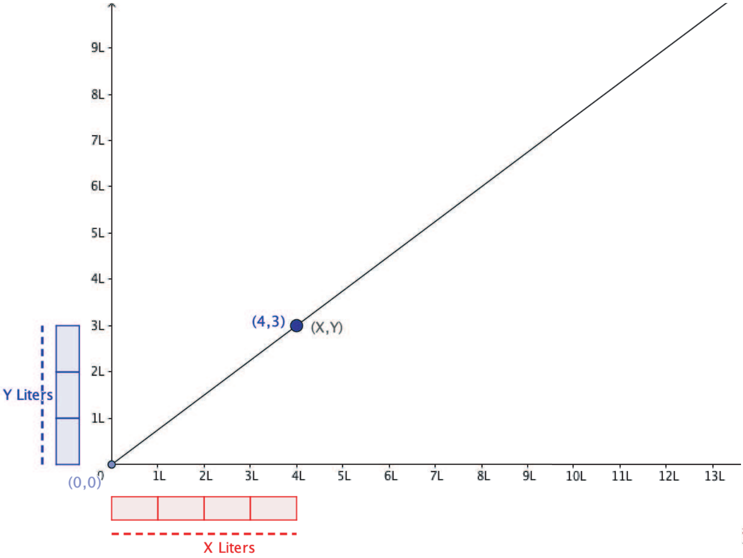

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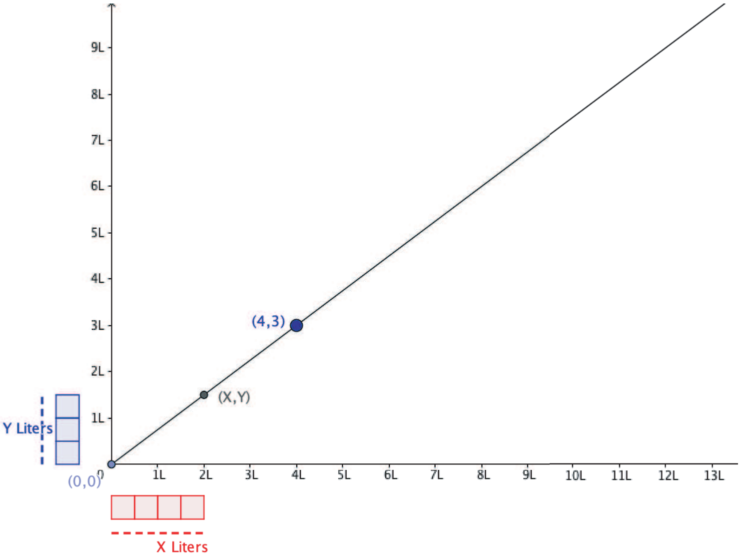

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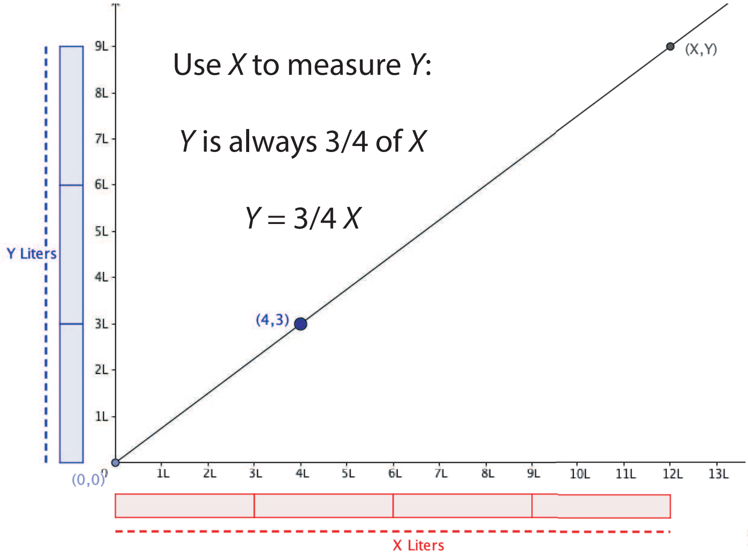

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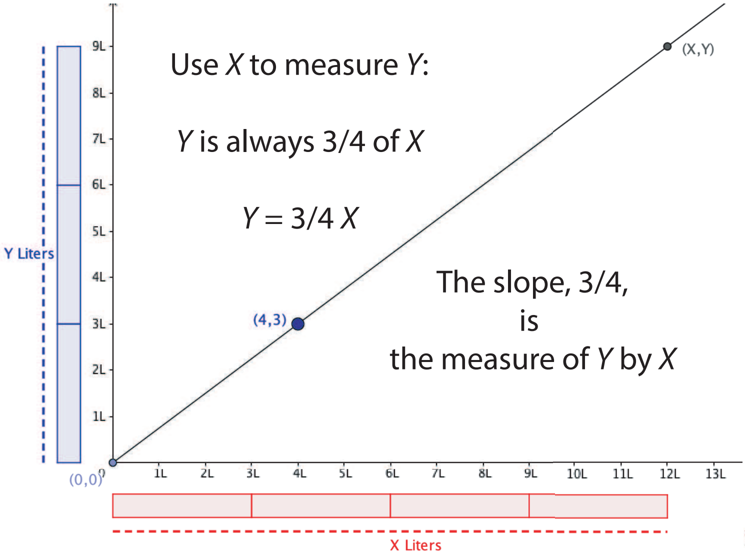

Use X to measure Y:

Y is always 3/4 of X

$Y = 3/4\,X$

The slope, 3/4,
is
the measure of Y by X

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Future teachers experience success

- Explaining why a problem is a multiplication or division problem
- Writing word problems for fraction addition, subtraction, multiplication, division
- Explaining why procedures for adding, subtracting, multiplying, dividing fractions are valid
- Deriving and explaining equations for linear relationships
- Etc!

My principle 3

Mathematics education is a collective, shared responsibility. Sharing the responsibility requires openness, compassion, learning with and from each other, trying ideas, and being willing to change our thinking.

Thank you!

Comments or questions?

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